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The path integral formulation of climate dynamics

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SUMMARY The chaotic nature of the atmospheric dynamics has stimulated the inclusions of methods and ideas derived from statistical dynamics. For instance, weather predictions have recently been based on the development of extensive ensemble systems that are designed to sample the phase space around the initial condition. Such an approach has been shown to improve substantially the usefulness of the forecasts allowing forecasters to issue probability-based forecasts. These works have modified the dominant paradigm of interpretation of the evolution of atmospheric flows (and to some extent also of oceanic motions) attributing more importance to the probability distribution of the variables of interest rather than to a single representation. The ensemble experiments can be considered as crude attempts to estimate the evolution of the probability distribution of the climate variables, that turns out to be the only physically meaningful quantity, but little work has been done on a direct modeling of the probability evolution itself. In this paper we show that it is possible to write the evolution of the probability distribution as a functional integral of the same kind introduced by Feynman in quantum mechanics, using some of the methods and results developed in statistical physics. The approach allows to obtain a formal solution to the Fokker-Planck equation corresponding to the Langevin-like equation of motion with noise. The method is very general and it provides a framework generalizable to red noise, lagged equations and even field equations, i.e. partial differential equations with noise. These concepts will be applied to an example taken from a simple model of ENSO.

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1 - INTRODUCTION

The equations that govern the evolution of the atmosphere and the ocean have been known for a long time and they have been extensively investigated. Numerical methods have been extensively developed and are today the approach of choice to solve them exploiting the first order time derivatives to obtain the time evolution. The equations soon showed a strong sensitivity to small perturbations, both in the initial conditions or in the parameters defining them, giving rise to the entire field of dynamical chaos [15].

The chaotic nature of the dynamics has stimulated the inclusions of methods and ideas derived from statistics and statistical dynamics. For instance, weather predictions have recently been based on the development of extensive ensemble systems that are designed to sample the phase space around the nominal initial condition. Such an approach has been shown to improve substantially the usefulness of the forecasts allowing forecasters to issue probability-based forecasts. The implicit assumption is that the presence of various sources of errors, coupled with the intrinsic sensitivity of the evolution equations to small errors [15], makes a single forecast less meaningful [4, 14].

The concept has gained a large consensus because it has been shown to be relevant to various dynamical problems. Numerical experiments driven by external forcing, like those used with prescribed SST or even prescribed concentration of greenhouses gases in climate change experiments, have shown that the response to external forcing is still sensitive to error growth, either because of uncertainties in initial condition or in model formulation. Ensemble experiments are now commonly used in these cases [23, 27].

These works have been modifying the dominant paradigm of interpretation of the evolution of atmospheric flows (and to some extent also of the ocean, see [21]) attributing more and more importance to the probability distribution of the variables of interest rather than to a single representation. The ensemble experiments can be considered as crude attempts to estimate the evolution of the probability distribution of the climate variables, that turns out to be the only physically meaningful quantity. Other interesting quantities can be obtained by computing expectation values over the PDF itself. Ensemble mean of temperature, for instance, cannot be considered simply as the average of the available ensemble members, but as the simplest estimation of the expectation value.

Finding an equation for the evolution of the PDF is far from being trivial. There are extensive works on the modification of the deterministic climate/weather evolution equations with a stochastic component [9] that have shown that a stochastic component is not in contradiction with the basic principles of the atmospheric/ocean dynamics and that such models can be considered to describe some aspects of the atmosphere correctly in mechanistic models [5, 2, 20, 3, 24, 26] and also estimating the stochastic component from observations [12, 7].

The addition of a stochastic noise to the evolution equation result in a multidimensional Langevin-like equation can be shown to support a Fokker-Planck equation for the evolution of the probability distribution of the state vector. This result is very interesting since the Fokker-Planck equation is linear, even if the corresponding evolution equation may be nonlinear. However, this approach, when applied directly to the discretized equation, is unpractical because the Fokker Planck equation is obtained in a phase space with the dimensions corresponding to the number of degrees of freedom of the original equation. Even a very simple general circulation model can easily have hundreds of degrees of freedom.

In this paper we show that it is possible to write the evolution of the probability distribution as a functional integral of the same kind introduced by Feynman [6] in quantum mechanics, using some of the methods and results developed in statistical physics [18, 8]. The approach allows to obtain a formal solution to the Fokker-Planck equation corresponding to the Langevin-like equation of motion with noise. The method is very general and it provides a framework easily generalizable to red noise, lagged equations and even field equations, i.e. partial differential equations with noise. The approach has been proved useful in fields other than physics, such as polymer theory, chemistry and even financial markets [22, 28, 11], but it has never been applied to atmospheric and oceanic problems and it is relatively less known in our community. The first quantum field theory formalism describing additive noise was developed by Martin, Siggia and Rose [16], but they have used a different kind of approach, a method similar to the canonical quantization.

In order to perform the perturbative expansion of correlation functions, we note that it is necessary to extend the Hubbard-Stratonovich transformation to the multidimensional case as well as the transformation introduced by Muñoz to calculate the path integral of the type considered.

After the introduction, we will introduce and summarize the general theoretical foundation in Section 2 and discuss the calculation of the integrals in Section 3.The concept of Green's matrix and functions will be introduced in Section 4 and a discussion of perturbation expansion applied to nonlinear cases will be introduced in Section 5.These concepts will be applied to an example taken from a simple model of ENSO [10] in Section 6 and the conclusions in Section 7 will close the paper.

2 - THE PATH INTEGRAL FORMULATION

2.1 - LANGEVIN EQUATION AND PROBABILITY

The systems describing the atmosphere or the ocean can be written as a Langevin equation:

$$\dot{q}_\mu(t) = f_\mu(\mathbf{q}(t)) + \epsilon_\mu(t) \tag{1}$$

where $\mathbf{q}(t) = (q_1(t), \ldots, q_K(t))$ represents a trajectory in $\mathbb{R}^K$ and $f_\mu(\mathbf{q})$ a differentiable function of $\mathbf{q}$. We are assuming that we have K degrees of freedom. The noise functions $\epsilon_\mu(t)$ are defined by their correlation functions as

$$\gamma_{\mu\nu}(t,t') =< \epsilon_\mu(t)\epsilon_\nu(t') >_\epsilon = Q\delta_{\mu\nu}\delta(t-t') \tag{2}$$

and they have zero means, $<>_\epsilon$ is an average with respect to the probability distribution of the realizations of the stochastic variables $\epsilon_\mu(t)$. A discretization can be applied, for instance denoting as t_0 and T the initial and final times we have

$$
\begin{aligned}
\tau &= (T - t_0)/N \\
t_n &= t_0 + n\tau \\
x_n &= x(t_n)
\end{aligned} \tag{3}
$$

if we integrate the Langevine equation (1) in an infinitesimal time interval τ, after integration of the noise, the discretized equation becomes

$$
\mathbf{q}_{n+1} - \mathbf{q}_n = \tau \mathbf{f}(\mathbf{q}_n) + \sqrt{\tau}\epsilon_n \tag{4}
$$

the probability distribution of the discretized noise is given by

$$
P(\epsilon_n) = (2\pi Q)^{-1/2} \exp\left(-\frac{\epsilon_n^2}{2Q}\right).
$$

the mistake we make in the discretization of the equations of this type is

$$
|\mathbf{q}(t_n + \tau) - \mathbf{q}(t_n)| = O(\sqrt{\tau}) \qquad \text{with} \qquad \tau \to 0. \tag{5}
$$

We can then write the conditional probability that the system will be in the state $\mathbf{q}_{n+1}$ at time t_{n+1} given that it was in $\mathbf{q}_n$ at time t_n,

$$
\begin{aligned}
p_n(\mathbf{q}_{n+1}, t_{n+1}|\mathbf{q}_n, t_n) = \int \delta(\mathbf{q}_{n+1} - \mathbf{q}_n - \tau\mathbf{f}(\mathbf{q}_n) - \sqrt{\tau}\epsilon_n)P(\epsilon_n)d\epsilon_n = \\
\frac{1}{(2\pi Q\tau)^{-1/2}} \exp\left[-\frac{(\mathbf{q}_{n+1} - \mathbf{q}_n - \tau\mathbf{f}(\mathbf{q}_n))^2}{2Q\tau}\right]
\end{aligned}
$$

using the Kolmogorov relation repeatedly

$$
p_n(\mathbf{q}_{n+2}, t_{n+2}|\mathbf{q}_n, t_n) = \int p_n(\mathbf{q}_{n+2}, t_{n+2}|\mathbf{q}_{n+1}, t_{n+1}) p_n(\mathbf{q}_{n+1}, t_{n+1}|\mathbf{q}_n, t_n) d\mathbf{q}_{n+1}
$$

we can obtain the probability for the entire path

$$
p_n(\mathbf{q}_T, T|\mathbf{q}_0, t_0) = \int \frac{d\mathbf{q}_1 \cdots d\mathbf{q}_{N-1}}{(2\pi Q\tau)^{N/2}} \exp\left(-\frac{S_N(\mathbf{q}_0, \ldots, \mathbf{q}_N)}{2Q}\right) \tag{6}
$$

where we have defined

$$
S_N(\mathbf{q}_0, \ldots, \mathbf{q}_N) = \sum_{n=0}^{N-1} \left[\frac{(\mathbf{q}_{n+1} - \mathbf{q}_n - \tau\mathbf{f}(\mathbf{q}_n))^2}{\tau}\right] \tag{7}
$$

The S_N functional plays the role of the action as in classical mechanics and it is also known as the Onsager-Machlup functional. There are N-1 integrations over the possible intermediate values of the path, but the end points $\mathbf{q}_0$ and $\mathbf{q}_T$ are fixed. Note that there are N factors in the denominator and so presumably we will have to introduce a normalization factor later.

2.2 - THE PROPAGATOR

The probability of reaching q_T, T from any point is then given by

$$p_n(\mathbf{q}_T, T) = \int \frac{d\mathbf{q}_1 \cdots d\mathbf{q}_{N-1}}{(2\pi Q\tau)^{N/2}} \exp\left(-\frac{S_N(\mathbf{q}_0, \ldots, \mathbf{q}_N)}{2Q}\right) p(\mathbf{q}_0, t_0) d\mathbf{q}_0 \tag{8}$$

that describes the evolution of the probability distribution from time t_0 to time T. It is practically the solution to the Fokker-Planck equation. The final integration on $\mathbf{q}_0$ resolves the normalization issues mentioned before and we obtain a finite result. We can also write (8) as

$$p_n(\mathbf{q}, t) = \int K_n(\mathbf{q}, t; \mathbf{q}_0, t_0) P_n(\mathbf{q}_0, t_0) d\mathbf{q}_0 \tag{9}$$

where we have introduced

$$K_n(\mathbf{q}, t; \mathbf{q}_0, t_0) = \int \frac{d\mathbf{q}_1 \cdots d\mathbf{q}_{N-1}}{(2\pi Q\tau)^{N/2}} \exp\left(-\frac{S_N(\mathbf{q}_0, \ldots, \mathbf{q}_N)}{2Q}\right) \tag{10}$$

that is a kernel that propagates the solution from time to time. It is known as the propagator.

The concept of the path integrals recurring in these formulas is illustrated in Fig. (1). The probability of reaching $\mathbf{q}_T$ starting at $\mathbf{q}_0$ is composed by the sum of all paths that may take all possible intermediate values at intermediate times. Their contribution must be integrated for all possible values.

It is also possible to formulate also a continuous time treatment. Going back to the Langevin equation:

$$\dot{q}_\mu(t) = f_\mu(\mathbf{q}(t)) + \epsilon_\mu(t) \tag{11}$$

we can formally define a probability distribution for the stochastic process that is a solution of the equation

$$P(\mathbf{q}, t; \mathbf{q}_0, t_0) = \left\langle \prod_{\mu=1}^{N} \delta[q_\mu(t) - q_\mu] \right\rangle_\epsilon \qquad \text{with} \qquad t \geq 0, \tag{12}$$

in which $\mathbf{q}_0$ and t_0 are our initial conditions. This probability is just the ensemble average over the solutions of the Langevin equation (1). We can use the probability definition (12) to define expectation values for all functions of $\mathbf{q}(t) = (q_1(t), q_2(t), \ldots, q_\mu(t))$,

$$\langle F(\mathbf{q}(t)) \rangle = \frac{\int P(\mathbf{q}, t; \mathbf{q}_0, t_0) F(\mathbf{q}(t)) dq_1 dq_2 \ldots dq_\mu}{\int P(\mathbf{q}, t; \mathbf{q}_0, t_0) dq_1 dq_2 \ldots dq_\mu} \tag{13}$$

.

In what follows we shall often omit the dependence on the initial data and use the simplified notation

$P(\mathbf{q}, t)$.

Using the gaussian nature of the noise, it is possible to write a Fokker-Planck equation for $P(\mathbf{q}, t)$ (summation is implied over repeated indices):

$$\frac{\partial P(\mathbf{q}, t)}{\partial t} = \frac{\partial}{\partial q_\mu}\left[\frac{1}{2}Q\frac{\partial P}{\partial q_\mu} - f_\mu(\mathbf{q})P\right]. \tag{14}$$

The solution of this equation can be written as a path integral [8]

$$P(\mathbf{q}, t) = \int [\mathcal{D}\mathbf{q}(\tau)] \exp\left(-S(\mathbf{q})\right) P(\mathbf{q}, t_0) \tag{15}$$

where the integration is done over all paths $\mathbf{q}(t)$ that go from t_0 to T. The functional $S(\mathbf{q})$ is the continuous Onsager-Machlup action that can then be defined in the general case as

$$S(\mathbf{q}) = \frac{1}{2}\int_0^T \left[[\dot{q}_\mu - f_\mu(\mathbf{q})]\gamma^{-1}_{\mu\nu}[\dot{q}_\nu - f_\nu(\mathbf{q})] + \frac{\partial f_\mu}{\partial q_\mu}\right] dt \tag{16}$$

and in the white noise case

$$S(\mathbf{q}) = \frac{1}{2Q}\int_0^T \left[[\dot{q}_\mu - f_\mu(\mathbf{q})]\delta_{\mu\nu}[\dot{q}_\nu - f_\nu(\mathbf{q})] + Q\frac{\partial f_\mu}{\partial q_\mu}\right] dt \tag{17}$$

The extra divergence term in the action is generated by the difficulty of defining the derivative of a stochastic process.

From equation (5) we note that the process paths, which are solutions of the Langevin equation, are therefore continuous as $\Delta t \to 0$, but they are not differentiable and therefore the ordinary rules of calculus must be modified to come up with a consistent definition. In the case of a simple additive noise the pathologies do not show up, but if we have multiplicative terms containing the noise then it is absolutely necessary to choose an interpretation. In the following we will be using the Stratonovich interpretation that allows to treat the fields as differentiable and therefore to use them in the ordinary rules of calculus. In the case of weak additive noise the Ito interpretation is also appropriate [8] and the divergence term drops, simplifying the action:

$$S_{weak}(\mathbf{q}) = \frac{1}{2Q}\int_0^T \left[[\dot{q}_\mu - f_\mu(\mathbf{q})]\delta_{\mu\nu}[\dot{q}_\nu - f_\nu(\mathbf{q})]\right] dt \tag{18}$$

The expression for the probability in the continuous case is given by

$$P(\mathbf{q}, t) = \int K(\mathbf{q}, t; \mathbf{q}_0, t_0) P(\mathbf{q}_0, t_0) d\mathbf{q}_0 \tag{19}$$

where $d\mathbf{q} = dq_1 dq_2 \ldots dq_\mu$ and continuous time propagator from time t_0 to t is

$$K(\mathbf{q}, t; \mathbf{q}_0, t_0) = \int [\mathcal{D}\mathbf{q}(\tau)] \exp(-S(\mathbf{q})) \tag{20}$$

Eq. (19) is the probability of finding the system in the state $\mathbf{q}$ at time t given that it was given by the distribution $P(q_0, t_0)$ at time t_0, but in this expression the path integral is now for paths with fixed extremes $\mathbf{q}_0 = \mathbf{q}(t_0), \mathbf{q} = \mathbf{q}(T)$.

2.3 - EXPECTATION VALUES AND CORRELATIONS

The expected value of some functions at a specific time, $0 < t_1 < t$, is given by (13) that can also be written as

$$< F(\mathbf{q}(t_1)) >= \mathcal{N} \int F(\mathbf{q}_1) \int [\mathcal{D}\mathbf{q}(\tau)] \exp(-S(\mathbf{q})) P(\mathbf{q}_0, t_0) d\mathbf{q}_0 d\mathbf{q}_1 \tag{21}$$

($\mathcal{N}$ is a normalization constant). This is a path integral that is now summing all the paths that go from $\mathbf{q}_0, t_0$ and arrive at $\mathbf{q}(t), t$ subject to the condition that they go through $\mathbf{q}_1, t_1$, so we can write (20) as

$$\langle F(\mathbf{q}(t_1)) \rangle = \mathcal{N} \int K(\mathbf{q}, t; \mathbf{q}_1, t_1) F(\mathbf{q}_1) K(\mathbf{q}_1, t_1; \mathbf{q}_0, t_0) P(\mathbf{q}_0, t_0) d\mathbf{q}_0 d\mathbf{q}_1 \tag{22}$$

but practically we are summing over all paths, subject to the conditions that they acquire the value $\mathbf{q}_1$ at t_1, so we can express the whole expectation value with a single path integral introducing the factor $F(\mathbf{q}(t_1))$ and making explicit the normalization,

$$\langle F(\mathbf{q}(t_1)) \rangle = \frac{\int [\mathcal{D}\mathbf{q}(\tau)] F(\mathbf{q}(t_1)) \exp(-S(\mathbf{q})) P(\mathbf{q}_0, t_0) d\mathbf{q}_0}{\int [\mathcal{D}\mathbf{q}(\tau)] \exp(-S(\mathbf{q})) P(\mathbf{q}_0, t_0) d\mathbf{q}_0} \tag{23}$$

Correlation can be obtained by choosing for F polynomial expressions $q_{i_1}(t_1) q_{i_2}(t_2) \ldots$ we get

$$\begin{aligned} &\langle q_{i_1}(t_1) q_{i_2}(t_2) \cdots, q_{i_{n-1}}(t_{n-1}) q_{i_n}(t_n) \rangle = \\ &\qquad \frac{\int [\mathcal{D}\mathbf{q}(\tau)] q_{i_1}(t_1) q_{i_2}(t_2) \ldots, q_{i_{n-1}}(t_{n-1}) q_{i_n}(t_n) \exp(-S(\mathbf{q})) P(\mathbf{q}_0, t_0) d\mathbf{q}_0}{\int [\mathcal{D}\mathbf{q}(\tau)] \exp(-S(\mathbf{q})) P(\mathbf{q}_0, t_0) d\mathbf{q}_0} \end{aligned} \tag{24}$$

3 - CALCULATING THE PATH INTEGRAL

The expression of the path integral is a formal expression and in order to give some meaning to it we have to treat it as the value of a limiting procedure through the discretization. We illustrate the concept using the weak noise action for simplicity. Using the discretization introduced in the previous Section we have to perform an integration at every point j as we are summing over all possible paths, i.e. functions, that interpolate between t_0 and T. In this way we obtain an approximate value of the path integral for a certain finite N. The exact value is then taken by the following limit, if it exists

$$K(\mathbf{q},t;\mathbf{q}_0,t_0)=\lim_{\substack{N\to\infty\\ N\tau\to t}}\int\cdots\int\mathcal{D}^d[\mathbf{q}(t)]\exp\left(-\frac{S(\mathbf{q}_j,\mathbf{q}_{j-1})}{2}\right) \tag{25}$$

and and the boundary conditions $\mathbf{q}_N=\mathbf{q}(T)$ and $\mathbf{q}_0=\mathbf{q}(t_0)$. The measure is introduced by integrating on the d-dimensional space at every point in the discretization j,

$$\mathcal{D}^d[\mathbf{q}(t)]=\frac{1}{(2Q\pi\tau)^{Nd/2}}\prod_{j=1}^{N-1}(dq_1\cdots dq_d)_j \tag{26}$$

and the discretized action is then

$$S(\mathbf{q}_j)=\frac{1}{\tau Q}\sum_{j=1}^{N}[\mathbf{q}_j-\mathbf{q}_{j-1}-\tau f_j(\mathbf{q}_{j-1})]^2 \tag{27}$$

The value of the path integral is then given by the existence of the infinite integration implicit in this relation. The choice of the discretization is important because the term

$$(\mathbf{q}_j-\mathbf{q}_{j-1})\mathbf{f}_j(\mathbf{q}_{j-1})$$

is ill-defined and it must be treated carefully. It turns out that Feynman's [6] original choice of symmetrizing the term as

$$(\mathbf{q}_j-\mathbf{q}_{j-1})\,\frac{\mathbf{f}(\mathbf{q}_{j-1})+\mathbf{f}(\mathbf{q}_j)}{2}$$

is equivalent to choosing the Stratonovich interpretation and the calculation is possible.

Practically, computable path integrals are rare and they are essentially limited to gaussian integrals. They can be calculated directly from the discretization introduced previously and it is definitely the method that shows the most delicate points and possible traps for the definition of the integrals. There is also another approach that can be used for quadratic lagrangians that is somewhat quicker that is based on the projection on eigenfunctions.

Gaussian path integrals are generated when the function $\mathbf{f}(q_j)$ is a linear operator $\mathbf{A}$. In this case the action can be written as

$$S(\mathbf{q})=\frac{1}{2Q}\int_0^T\left[[\dot{\mathbf{q}}-\mathbf{A}\mathbf{q}]^T[\dot{\mathbf{q}}-\mathbf{A}\mathbf{q}]\right]+QTr(\mathbf{A})dt \tag{28}$$

and the path integrals become

$$K(\mathbf{q},T;\mathbf{q}_0,t_0)=e^{\left(-\frac{1}{2}Tr(\mathbf{A})T\right)}\int[\mathcal{D}\mathbf{q}(\tau)]\exp(-\frac{1}{2Q}\int_0^T[\dot{\mathbf{q}}-\mathbf{A}\mathbf{q}]^T[\dot{\mathbf{q}}-\mathbf{A}\mathbf{q}]dt) \tag{29}$$

The usual procedure requires here that we seek stationarity conditions for the action. However, there is a problem generated by the fact that for a system of the present form there are two solutions to the first order variations that correspond to the equation of motion. The solutions correspond to the choice $\mathbf{q} = \mathbf{r}$ such that

$$\dot{\mathbf{r}} = \pm \mathbf{A}\mathbf{r} \qquad \mathbf{r}(0) = \mathbf{q}_0$$

the unperturbed trajectory corresponds to the plus sign and obviously it would be desirable to be able to investigate the perturbation around this solution, but this is tricky because the particular value of the action in this case is zero making a traditional expansion impossible. However, as it was pointed out by [17] there is a method that allows the expansion along the correct solution and also satisfies both boundary conditions for the integration in the action. It is necessary to introduce a change of variables quantity $\mathbf{q} = \mathbf{r} + \mathbf{g}$, such that the action (28) can be written as

$$S = -\frac{1}{2Q}\int_0^T \left(\dot{\mathbf{r}} + \dot{\mathbf{g}} - \mathbf{A}(\mathbf{r}+\mathbf{g})\right)^T \left(\dot{\mathbf{r}} + \dot{\mathbf{g}} - \mathbf{A}(\mathbf{r}+\mathbf{g})\right) dt = -\frac{1}{2Q}\int_0^T \left(\dot{\mathbf{g}} - \mathbf{A}\mathbf{g}\right)^T \left(\dot{\mathbf{g}} - \mathbf{A}\mathbf{g}\right) dt \quad (30)$$

because the $\mathbf{r}$ satisfies the equation of motion. The boundary conditions on this expression are given by

$$\mathbf{g}(0) = 0 \qquad \mathbf{g}(T) = \mathbf{q}_T - \mathbf{r}(T)$$

The measure of the integral does not change since it is a linear transformation. We can now substitute around an unperturbed trajectory such that deviations of order $\sqrt{Q}$ are introduced obeying the boundary conditions $\mathbf{y}(0) = \mathbf{y}(T) = 0$

$$\mathbf{g}(t) = \mathbf{g}_c(t) - \mathbf{y}(t)\sqrt{Q} \quad (31)$$

Substituing eq. (31) in the Action (29) we get

$$S = \int_0^T \left(\dot{\mathbf{g}_c} - \mathbf{A}\mathbf{g}_c\right)^T \left(\dot{\mathbf{g}_c} - \mathbf{A}\mathbf{g}_c\right) + 2\left(\dot{\mathbf{y}} - \mathbf{A}\mathbf{y}\right)^T \left(\dot{\mathbf{g}_c} - \mathbf{A}\mathbf{g}_c\right) + \left(\dot{\mathbf{y}} - \mathbf{A}\mathbf{y}\right)^T \left(\dot{\mathbf{y}} - \mathbf{A}\mathbf{y}\right) dt \quad (32)$$

integrating by parts the various terms and using the boundary conditions we obtain

$$\begin{gathered} S = -\tfrac{1}{2Q}\left(\mathbf{g}_c^T \dot{\mathbf{g}_c} + \mathbf{g}_c^T \mathbf{A}\mathbf{g}_c\right)|_0^T - \\ \tfrac{1}{2Q}\int_0^T \mathbf{g}_c^T \left(-\ddot{\mathbf{g}_c} + (\mathbf{A}^T - \mathbf{A})\dot{\mathbf{g}_c} + \mathbf{A}\mathbf{A}^T\mathbf{g}_c\right) dt - \\ \tfrac{1}{2Q}\int_0^T \mathbf{y}^T \left(-\ddot{\mathbf{g}_c} + (\mathbf{A}^T - \mathbf{A})\dot{\mathbf{g}_c} + \mathbf{A}\mathbf{A}^T\mathbf{g}_c\right) dt - \\ \tfrac{1}{2Q}\int_0^T \mathbf{y}^T \left(-\ddot{\mathbf{y}} + (\mathbf{A}^T - \mathbf{A})\dot{\mathbf{y}} + \mathbf{A}\mathbf{A}^T\mathbf{y}\right) dt - \end{gathered} \quad (33)$$

so if we choose a $\mathbf{g}_c$ that satisfies the equation with the given boundary conditions

$$-\ddot{\mathbf{g}_c} + (\mathbf{A}^T - \mathbf{A})\dot{\mathbf{g}_c} + \mathbf{A}\mathbf{A}^T\mathbf{g}_c = 0 \tag{34}$$

we can divide the action in two parts, the explicit terms depending on the boundary conditions and implicitly from the unperturbed solution $\mathbf{r}$ and a term that depends only on the fluctuations $\mathbf{y}$,

$$\begin{aligned} S = -\tfrac{1}{2Q}\left(\mathbf{g}_c^T\dot{\mathbf{g}_c} + \mathbf{g}_c^T\mathbf{A}\mathbf{g}_c\right)\big|_0^T - \\ \tfrac{1}{2Q}\int_0^T \mathbf{y}^T\left(-\ddot{\mathbf{y}} + (\mathbf{A}^T - \mathbf{A})\dot{\mathbf{y}} + \mathbf{A}\mathbf{A}^T\mathbf{y}\right)dt \quad &= S_1 + S_2 \end{aligned}$$

The term S_1 does not depend on the varying path and therefore can be taken out from the integration whereas the term S_2 will depend only on time and it is often called the *prefactor*. The propagator (29) can then be written as

$$\begin{aligned} K(\mathbf{q}, T; \mathbf{q}_0, t_0) &= \exp\left(-\frac{1}{2}Tr(\mathbf{A})T\right) \\ &\exp\left(-\frac{S_1}{2Q}\right)\int[\mathcal{D}y(\tau)]\exp\left(-\frac{1}{2Q}\int_0^T \mathbf{y}^T\left(-\ddot{\mathbf{y}} + (\mathbf{A}^T - \mathbf{A})\dot{\mathbf{y}} + \mathbf{A}\mathbf{A}^T\mathbf{y}\right)dt\right) \end{aligned} \tag{35}$$

with boundary conditions $y(0) = y(T) = 0$

The remaining calculation can be finished by observing that the action in the paths y is then equivalent to a Sturm-Liouville boundary problem for the differential operator $\mathbf{\Lambda}$

$$\int_0^T \mathbf{y}^T\left(-\ddot{\mathbf{y}} + (\mathbf{A}^T - \mathbf{A})\dot{\mathbf{y}} + \mathbf{A}\mathbf{A}^T\mathbf{y}\right)dt = \int_0^T \left[\mathbf{y}^T\mathbf{\Lambda}\mathbf{y}\right]dt \tag{36}$$

The operator $\mathbf{\Lambda}$ is self-adjoint and so it has a complete orthonormal set of eigenfunctions $\phi_{n_1,n_2,\ldots,n_d}$ with real eigenvalues $\mu_{n_1}, \mu_{n_2}, \ldots, \mu_{n_d}$. The eigenfunction and eigenvalues are d-multiple infinities as a consequence of the dimensionality d of the operator. We can expand the variables $\mathbf{y}$ in series of the complete orthonormal eigenfunctions and we get

$$\frac{1}{2Q}\int_0^T \mathbf{y}^T\mathbf{\Lambda}\mathbf{y}dt = \frac{1}{2Q}\int_0^T \sum_{n_i} c_{n_i}\phi_{n_i}\sum_{n_j}\mu_{n_j}c_{n_j}\phi_{n_j} = \frac{1}{2Q}\sum_{n_i} c_i^2\mu_{n_i} \tag{37}$$

Using this approach we can write the path integral (20) as an infinite set of gaussian integrals over the coefficients of the expansion. A change of variables from the variables from the q to the c will allow the execution of the integral. Because Λ is self-adjoint it can be diagonalized by a unitary transformation with a unit Jacobian for the change of variables so the path integral measure remains the same,

$$K(\mathbf{q},T;\mathbf{q}_0,t_0) = \exp\left(-\frac{1}{2Q}S_1\right)\int\cdots\int \frac{1}{(2Q\pi)^{Nd/2}}\prod_{j=1}^{N-1}(dc_1\cdots dc_d)_j \exp\left(-\frac{1}{2Q}\sum_{i=1}^{N-1}c_i^2\mu_{n_i}\right) \quad (38)$$

and the boundary conditions are satisfied by the eigenfunctions. The integral is then formed by an infinite number of gaussian integrals that each contributes a factor

$$K(\mathbf{q},T;\mathbf{q}_0,t_0) = e^{\left(-\frac{1}{2}Tr(\mathbf{A})T\right)}\exp\left(-\frac{1}{2Q}S_1\right)\frac{1}{(2Q\pi)^{Nd/2}}\prod_{n_i=1,i=1}^{N-1,d}\left(\frac{2\pi Q}{\mu_{n_i}}\right)^{1/2} \quad (39)$$

or

$$K(\mathbf{q},T;\mathbf{q}_0,t_0) = e^{\left(-\frac{1}{2}Tr(\mathbf{A})T\right)}\exp\left(-\frac{1}{2Q}S_1\right)\frac{1}{2Q\pi}\prod_{n_i=1,i=1}^{N-1,d}\left(\frac{1}{\mu_{n_i}}\right)^{1/2} \quad (40)$$

The case of the Free evolution of the system with $A=0$ is interesting and it corresponds to a pure Brownian motion. In this case we get

$$K_F(\mathbf{q},T;\mathbf{q}_0,t_0) = \exp\left(-\frac{1}{2Q}S_1^F\right)\frac{1}{2Q\pi}\prod_{n_i=1,i=1}^{N-1,d}\left(\frac{T^2}{n_i^2\pi^2}\right)^{1/2} \quad (41)$$

but K_F can be obtained also by a direct calculation [25, 6] and we get

$$K_F(\mathbf{q},T;\mathbf{q}_0,t_0) = \left(\frac{1}{2Q\pi(T-t_0)}\right)^{d/2}\exp\left(-\frac{1}{2Q}\frac{(\mathbf{q}-\mathbf{q}_0)^2}{T-t_0}\right) = \left(\frac{1}{2Q\pi(T-t_0)}\right)^{d/2}\exp(-\frac{S_1^F}{2Q}) \quad (42)$$

The Free Propagator provides a convenient reference frame for the other cases as the ratio

$$\frac{K}{K_F} = e^{\left(-\frac{1}{2}Tr(\mathbf{A})T\right)}\exp\left[-\frac{1}{2Q}(S_1-S_1^F)\right]\left(\frac{\det\Lambda_F}{\det\Lambda}\right)^{1/2}$$

and using the explicit expression (42) we have

$$K(\mathbf{q},T;\mathbf{q}_0,t_0) = e^{\left(-\frac{1}{2}Tr(\mathbf{A})T\right)}\left(\frac{1}{2Q\pi(T-t_0)}\right)^{d/2}\exp\left[-\frac{1}{2Q}S_1\right]\left(\frac{\det\Lambda_F}{\det\Lambda}\right)^{1/2} \quad (43)$$

where

$$\det\Lambda_F = \prod_{n_i=1,i=1}^{N,d}\left(\frac{T^2}{n_i^2\pi^2}\right)$$

and

$$\det \Lambda = \prod_{n_i=1, i=1}^{N,d} (\mu_{n_i}) \,.$$

More general quadratic actions, including the case of time-varying coefficients, can also be computed completely [19].

4 - GREEN'S FUNCTIONS AND GENERATING FUNCTIONS

The stochastic process that is a solution to the Langevin equation (1) is completely determined by the normalized multi-points correlation functions

$$G^{(n)}_{\alpha_1 \ldots \alpha_\mu}(t_1, t_2, \ldots, t_n) = \frac{\left\langle q_{\alpha_1}(t_1) q_{\alpha_2}(t_2) \ldots, q_{\alpha_{n-1}}(t_{n-1}) q_{\alpha_n}(t_n) \right\rangle}{N}$$

the indices α count the different variables in a multi-dimensional case. These functions express the probability that the path assumes the values $q_{\alpha_1}(t_1) q_{\alpha_2}(t_2) \ldots, q_{\alpha_{n-1}}(t_{n-1}) q_{\alpha_n}(t_n)$ at the respective times. It is also the expectation value for the same polynomial products and so it is the basic building block for the series expansion of any other functional.

The n-point function can be normalized as

$$G^{(n)}_{\alpha_1 \ldots \alpha_\mu}(t_1, t_2, \ldots, t_n) = \frac{\int [\mathcal{D}\mathbf{q}(\tau)] q_{\alpha_1}(t_1) q_{\alpha_2}(t_2) \ldots, q_{\alpha_{n-1}}(t_{n-1}) q_{\alpha_n}(t_n) \exp(-S(\mathbf{q}))}{\int [\mathcal{D}\mathbf{q}(\tau)] \exp(-S(\mathbf{q}))}$$

where we have used $P(\mathbf{q}_0, t_0) = \delta(\mathbf{q} - \mathbf{q}_0)$ for simplicity.

The calculation of the n-points correlation functions is tricky, but it can be simplified by introducing a special functional, the moment generating functional

$$Z[J] = \frac{\int [\mathcal{D}\mathbf{q}(\tau)] \exp\left(-S(\mathbf{q}) + \int \mathbf{J}(t) \cdot \mathbf{q}(t) dt\right)}{\int [\mathcal{D}\mathbf{q}(\tau)] \exp(-S(\mathbf{q}))} \tag{44}$$

the functional derivative of which

$$\left(\frac{\delta}{\delta J_\mu(t_1)} Z[J] \right) \bigg|_{J=0} = \frac{\int [\mathcal{D}\mathbf{q}(\tau)] q_\mu(t_1) \exp(-S(\mathbf{q}))}{\int [\mathcal{D}\mathbf{q}(\tau)] \exp(-S(\mathbf{q}))} = \langle q_\mu(t_1) \rangle \tag{45}$$

provides the expectation value for the mean. Repeating the process we can get the higher order correlations

$$\left(\frac{\delta}{\delta J_\mu(t_1)} \frac{\delta}{\delta J_\nu(t_2)} Z[J] \right) \bigg|_{\mathbf{J}=0} = \frac{\int [\mathcal{D}\mathbf{q}(\tau)] q_\mu(t_1) q_\nu(t_2) \exp(-S(\mathbf{q}))}{\int [\mathcal{D}\mathbf{q}(\tau)] \exp(-S(\mathbf{q}))} = \langle q_\mu(t_1) q_\nu(t_2) \rangle \tag{46}$$

and for a generic functional we have

$$\left(F[\frac{\delta}{\delta J_\mu(t_1)}]Z[J]\right)\Big|_{\mathbf{J}=0} = \frac{\int[\mathcal{D}\mathbf{q}(\tau)]F[q_\mu(t_1)]\exp(-S(\mathbf{q}))}{\int[\mathcal{D}\mathbf{q}(\tau)]\exp(-S(\mathbf{q}))} = \langle F[q_\mu(t_1)]\rangle\,. \tag{47}$$

The generating function can be obtained by evaluating the path integral

$$Z[J] = \frac{\int[\mathcal{D}\mathbf{q}(\tau)]\exp(-S(\mathbf{q}) + \int \mathbf{q}^T\mathbf{J}(t))dt}{\int[\mathcal{D}\mathbf{q}(\tau)]\exp(-S(\mathbf{q}))dt} \tag{48}$$

In the case of quadratic action the generating function can be calculated explicitly. We can start from the action in eq. (30) to get

$$Z[J] = \frac{\int[\mathcal{D}\mathbf{q}(\tau)]\exp\left(\int -\frac{1}{2Q}(\dot{\mathbf{g}} - A\mathbf{g})^T(\dot{\mathbf{g}} - A\mathbf{g}) + \mathbf{g}^T\mathbf{J}(t)\right)dt}{\int[\mathcal{D}\mathbf{q}(\tau)]\exp\left(\int -\frac{1}{2Q}(\dot{\mathbf{g}} - A\mathbf{g})^T(\dot{\mathbf{g}} - A\mathbf{g})\right)dt}. \tag{49}$$

We can look for deviation from the solution of the stationarity condition for the path integral given by the equation

$$-\ddot{\mathbf{g}}_c + (A^T - A)\dot{\mathbf{g}}_c + A^TA\mathbf{g}_c = Q\mathbf{J} \tag{50}$$

so that $\mathbf{g} = \mathbf{g}_c + \sqrt{Q}\mathbf{y}$. The numerator can be expressed as

$$\begin{aligned} Z[J] = \exp\left(\int -\frac{1}{2Q}\mathbf{g}_c^T(\ddot{\mathbf{g}}_c + (A^T - A)\dot{\mathbf{g}}_c + A^TA\mathbf{g}_c) + \mathbf{g}_c^T\mathbf{J}(t)\right) \\ \times \int[\mathcal{D}\mathbf{q}(\tau)]\exp\left(-\int(\dot{\mathbf{y}} - A\mathbf{y})^T(\dot{\mathbf{y}} - A\mathbf{y})\right)dt \\ \times \exp\left(\mathbf{g}_c^T(\dot{\mathbf{g}}_c - A\mathbf{g}_c)|_0^T\right) \end{aligned} \tag{51}$$

using the equation of motion (50) we then get

$$\begin{aligned} Z[J] = \exp\left(\int \frac{1}{2}\mathbf{g}_c^T\mathbf{J}(t)\right) \\ \times \int[\mathcal{D}\mathbf{q}(\tau)]\exp\left(-\int(\dot{\mathbf{y}} - A\mathbf{y})^T(\dot{\mathbf{y}} - A\mathbf{y})\right)dt \\ \times \exp\left(\mathbf{g}_c^T(\dot{\mathbf{g}}_c - A\mathbf{g}_c)|_0^T\right) \end{aligned} \tag{52}$$

and the normalized characteristic function is then simply

$$\frac{Z[J]}{Z[0]} = \exp\left(\int \frac{1}{2}\mathbf{g}_c^T(\mathbf{J}(t)\right)$$

The solution $\mathbf{g}_c$ can be obtained via the Green's function of the equation of motion

$$\left(\frac{d^2}{dt^2} + (A^T - A)\frac{d}{dt} - A^T A\right) G_A(\tau, \tau') = \delta(\tau - \tau') \tag{53}$$

with $G(0, \tau') = 0$, $G_A(T, \tau') = \mathbf{q}_T - \mathbf{r}(T)$ so

$$\mathbf{g}_c(\tau) = \int_0^T G_A(\tau, \tau')\mathbf{J}(\tau')d\tau'$$

and so the generating function is given by

$$Z[J] = \exp\left(\frac{1}{2}\int_0^T \int_0^T \mathbf{J}(\tau)G_A(\tau, \tau')\mathbf{J}(\tau')d\tau d\tau'\right)$$

5 - PERTURBATION EXPANSIONS

5.1 - FEYNMAN DIAGRAMS

The path integral formulation adapts itself very naturally to the definition of perturbation expansions of various kinds to compute the correction to the probability distribution and the correlation functions. The technique looks involuted, but it can be generalized very easily and it can be the base for applications to field equations arising in a field theory.

Consider the propagator for a nonlinear evolution, $\dot{\mathbf{q}} - \mathbf{A}\mathbf{q} - \mu\mathbf{f}(\mathbf{q}) = 0$, where μ is parameter that measures the strength of the nonlinear terms,

$$K(\mathbf{q}, T; \mathbf{q}_0, t_0) = \int_{\mathbf{q}_0}^{\mathbf{q}_T} [\mathcal{D}\mathbf{q}(\tau)] \exp\left(-\frac{1}{2Q}\int_0^T \left(\dot{\mathbf{q}} - \mathbf{A}\mathbf{q} - \mu\mathbf{f}\right)^T \left(\dot{\mathbf{q}} - \mathbf{A}\mathbf{q} - \mu\mathbf{f}(\mathbf{q})\right) dt\right) \tag{54}$$

We can introduce the same coordinate transformation described in Sect. 3 so that the action can be written as

$$S = -\frac{1}{2Q}\int_0^T \left(\dot{\mathbf{g}} - \mathbf{A}\mathbf{g} - \mu\mathbf{f}(\mathbf{r} + \mathbf{g})\right)^T \left(\dot{\mathbf{g}} - \mathbf{A}\mathbf{g} - \mu\mathbf{f}(\mathbf{r} + \mathbf{g})\right). \tag{55}$$

The quadratic nature of the action creates a potential problem because the expansion of the terms according to powers of the coupling constant μ generate terms of the form $\dot{\mathbf{g}}\mathbf{f}(\mathbf{g})$ that couples state variables with derivatives. It is possible to overcome this problem by using the Hubbard-Stratonovich transformation [13, 1], extended to the multidimensional case, that is a generalization of the identity to the functional integrals

$$\exp\left(-\frac{x^2}{2a}\right) = \sqrt{\frac{a}{2\pi}}\int_{-\infty}^{\infty} \exp\left(-\frac{ay^2}{2} - ixy\right) dy.$$

The propagator becomes then

$$K(\mathbf{g},T;\mathbf{g}_0,0)=\int\mathcal{D}[\mathbf{y}(t)]\int_{\mathbf{g}_0}^{\mathbf{g}_T}\mathcal{D}[\mathbf{g}(t)]$$
$$\exp\left(-\int_0^T\frac{Q\mathbf{y}^T\mathbf{y}}{2}-i\mathbf{y}^T\left(\dot{\mathbf{g}}-\mathbf{A}\mathbf{g}-\mu\mathbf{f}(\mathbf{g}+\mathbf{r})\right)+\frac{\mu}{2}\partial_i f_i(\mathbf{g}+\mathbf{r})+Tr(\mathbf{A})dt\right) \quad (56)$$

where the auxiliary functions $\mathbf{y}(t)$ are defined over the entire time axis. We can introduce the field $\boldsymbol{\phi}(t)=-i\mathbf{y}(t)$ and the trace of the linear part can be taken outside the functional integrals as it does not depend on the paths, yielding

$$K(\mathbf{g},T;\mathbf{g}_0,0)=\exp\left(-\int_0^T Tr(\mathbf{A})dt\right)$$
$$\int\mathcal{D}[\boldsymbol{\phi}(t)]\int_{\mathbf{g}_0}^{\mathbf{g}_T}\mathcal{D}[\mathbf{g}(t)]\exp\left[-\int_0^T\frac{Q\boldsymbol{\phi}^*\boldsymbol{\phi}}{2}+\boldsymbol{\phi}^*(\dot{\mathbf{g}}-\mathbf{A}\mathbf{g})dt\right]\times$$
$$\exp\left[\int_0^T\mu\boldsymbol{\phi}^*\mathbf{f}(\mathbf{g}+\mathbf{r})-\frac{\mu}{2}\partial_i\mathbf{f}(\mathbf{g}+\mathbf{r})\right]dt \quad (57)$$

or

$$K_V(\mathbf{g},T;\mathbf{g}_0,0)=\int\mathcal{D}[\boldsymbol{\phi}(t)]\int_{\mathbf{g}_0}^{\mathbf{g}_T}\mathcal{D}[\mathbf{g}(t)]\exp(-S_0)\exp(\int_0^T V(t)dt) \quad (58)$$

the subscript V has been added to underscore the dependence of this propagator on the nonlinear terms in the second exponential $\exp(V)$, whereas the quadratic terms are contained in S_0. The term V contains higher order terms that reflect the impact of the nonlinear interactions. The scalar product is defined as $(\mathbf{x},\mathbf{y})=\mathbf{x}^*\dot{\mathbf{y}}$ where the asterisk indicates hermitian conjugation, $(.^*)=\bar{(.)}^T$.

The propagator corresponding to the quadratic part describes the evolution of the system without interaction and so it can be described as the free evolution of the system. Usually it can be computed exactly:

$$K_0(\mathbf{g},T;\mathbf{g}_0,0)=\int\mathcal{D}[\phi]\int_{\mathbf{g}_0}^{\mathbf{g}_T}\mathcal{D}[\mathbf{g}(t)]\exp(-S_0) \quad (59)$$

whereas in the presence of interactions we have

$$K_V(\mathbf{g},T;\mathbf{g}_0,0)=\int\mathcal{D}[\phi]\int_{\mathbf{g}_0}^{\mathbf{g}_T}\mathcal{D}[\mathbf{g}(t)]\exp(-S_0)\exp\left(\int_0^T V(\mathbf{g},\phi)dt\right)=\left\langle\exp\left(\frac{1}{2Q}\int_0^T V(\mathbf{g},\phi)dt\right)\right\rangle_0 \quad (60)$$

in other words the propagator for the problem is the expected value of the interaction with respect to the probability distribution of the unperturbed, usually linear, problem.

In the presence of a small coupling constant the exponential for the interaction can be expanded in series, yielding successive correction to the free propagator

$$K_V(\mathbf{g}, T; \mathbf{g}_0, 0) = K_0 \left(1 + \frac{1}{2Q} \left\langle \int_0^T V(\mathbf{g}, \phi) dt \right\rangle_0 + \frac{1}{4Q^2} \left\langle \frac{1}{2} \int_0^T \int_0^T V(\mathbf{g}(t), \phi(t)) V(\mathbf{g}(t'), \phi(t')) dt dt' \right\rangle_0 + \cdots \right). \tag{61}$$

In the particular case of a polynomial interaction, the expectation values correspond to the moments of V with respect the unperturbed probability distribution. In the case of a polynomial V these expectation values can be computed using the generating functional (46).

5.2 - PERTURBATION EXPANSION FOR THE CORRELATION FUNCTIONS

The generating function can also be written for the nonlinear case using the transformed action (57). It is convenient to write it using using the real vector $\mathbf{J} = (\mathbf{j}, \mathbf{k}) = (j_1, j_2, k_3, k_4)$ as the source term, so that

$$Z(\mathbf{J}) = \frac{\int \mathcal{D}[\phi] \int_{\mathbf{g}_0}^{\mathbf{g}_T} \mathcal{D}[\mathbf{g}] \exp\left(- \int_0^T \frac{1}{2} Q \boldsymbol{\phi}^* \boldsymbol{\phi} + \boldsymbol{\phi}^* (\dot{\mathbf{g}} - \mathbf{A}\mathbf{g}) - \mathbf{g}^* \mathbf{j} - \boldsymbol{\phi}^* \mathbf{k} \, dt \right) \exp\left[\int_0^T V(\mathbf{g}, \mathbf{r}, \boldsymbol{\phi}) dt \right]}{\int \mathcal{D}[\phi] \int_{\mathbf{g}_0}^{\mathbf{g}_T} \mathcal{D}[\mathbf{g}(t)] \exp\left(- \int_0^T \frac{1}{2} Q \boldsymbol{\phi}^* \boldsymbol{\phi} + \boldsymbol{\phi}^* (\dot{\mathbf{g}} - \mathbf{A}\mathbf{g}) dt \right)} \tag{62}$$

where

$$\exp(\int_0^T V(\mathbf{g}, \mathbf{r}, \boldsymbol{\phi}) dt) = \exp\left[\int_0^T \mu \boldsymbol{\phi}^T \mathbf{f}(\mathbf{g} + \mathbf{r}) - \frac{\mu}{2} \partial_i \mathbf{f}(\mathbf{g} + \mathbf{r}) dt \right]$$

for a small coupling constant μ we can expand the exponential in a Taylor series to obtain

$$\exp(\int V(t) dt) = 1 + \mu \int V(t) dt + \frac{\mu^2}{2} \int \int V(t) V(t') dt dt' \tag{63}$$

When the function of the path V is a polynomial every term is the expectation value of the terms of the series expansion of the exponential and each can be obtained by differentiating the generating function of the free evolution. We can formally exponentiate the series and write for the generating function of the nonlinear case

$$Z(\mathbf{J}) = \exp\left(V(\frac{\delta}{\delta \mathbf{J}}) \right) Z_0(J) \tag{64}$$

that must be normalized by $Z(0)$. The expression for the quadratic generating function can be written as

$$Z_0(\mathbf{J}) = \frac{\int \mathcal{D}[\phi] \int_{\mathbf{g}_0}^{\mathbf{g}_T} \mathcal{D}[\mathbf{g}] \exp\left(-\int_0^T \frac{1}{2} Q \boldsymbol{\phi}^* \boldsymbol{\phi} + \boldsymbol{\phi}^* (\dot{\mathbf{g}} - \mathbf{A}\mathbf{g}) - \mathbf{g}^* \mathbf{j} - \boldsymbol{\phi}^* \mathbf{k} \, dt\right)}{\int \mathcal{D}[\phi] \int_{\mathbf{g}_0}^{\mathbf{g}_T} \mathcal{D}[\mathbf{g}(t)] \exp\left(-\int_0^T \frac{1}{2} Q \boldsymbol{\phi}^* \boldsymbol{\phi} + \boldsymbol{\phi}^* (\dot{\mathbf{g}} - \mathbf{A}\mathbf{g}) dt\right)} \tag{65}$$

where we have added a zero subscript to indicate that it is the generating function for the linear evolution. Introducing the vector $\mathbf{u} = (\mathbf{g}, \phi)$ we can write

$$Z_0(\mathbf{J}) = \frac{\int_{\mathbf{u}_0}^{\mathbf{u}_T} \mathcal{D}[\mathbf{u}(t)] \exp\left(-\int_0^T \frac{1}{2} \mathbf{u}^* \Delta^{-1} \mathbf{u} - \mathbf{u}^* \mathbf{J} \, dt\right)}{\int_{\mathbf{u}_0}^{\mathbf{u}_T} \mathcal{D}[\mathbf{u}(t)] \exp\left(-\frac{1}{2} \int_0^T \mathbf{u}^* \Delta^{-1} \mathbf{u} \, dt\right)} \tag{66}$$

where Δ^{-1} is the hermitian operator

$$\Delta^{-1} = \begin{pmatrix} 0 & -\partial_t + \mathbf{A}^* \\ \partial_t + \mathbf{A} & Q \end{pmatrix}. \tag{67}$$

As shown in (51) and the following analysis, we can get an explicit form for $Z_0[J]$ by inserting a shift $\mathbf{u} = \mathbf{u}_c + \mathbf{w}$ and the numerator becomes

$$Z_0[J] = \int_{\mathbf{w}_0}^{\mathbf{w}_T} \mathcal{D}[\mathbf{w}(t)] \exp\left(-\int_0^T \frac{1}{2} \mathbf{u}_c^* \Delta^{-1} \mathbf{u}_c + \frac{1}{2} \mathbf{w}^* \Delta^{-1} \mathbf{u}_c + \frac{1}{2} \mathbf{u}_c^* \Delta^{-1} \mathbf{w} + \frac{1}{2} \mathbf{w}^* \Delta^{-1} \mathbf{w} - \mathbf{u}_c^* \mathbf{J} - \mathbf{w}^* \mathbf{J} \, dt\right) \tag{68}$$

and we can find $\mathbf{u}_c$ so that $\Delta^{-1}\mathbf{u}_c - \mathbf{J} = 0$ then

$$\begin{aligned} Z_0[J] = \int_{\mathbf{w}_0}^{\mathbf{w}_T} \mathcal{D}[\mathbf{w}(t)] \exp\left(-\int_0^T \frac{1}{2} \mathbf{u}_c^* \mathbf{J} + \mathbf{w}^* \mathbf{J} + \frac{1}{2} \mathbf{w}^* \Delta^{-1} \mathbf{w} - \mathbf{u}_c^* \mathbf{J} - \mathbf{w}^* \mathbf{J} \, dt\right) = \\ \int_{\mathbf{u}_0}^{\mathbf{u}_T} \mathcal{D}[\mathbf{w}(t)] \exp\left(\int_0^T \frac{1}{2} \mathbf{u}_c^* \mathbf{J} \, dt\right) \exp\left(-\frac{1}{2} \mathbf{w}^* \Delta^{-1} \mathbf{w} \, dt\right) \end{aligned} \tag{69}$$

the remaining path integral over $\mathbf{w}(t)$ is eliminated by the normalization and so the generating function is given by

$$Z_0[J] = \exp\left(\int_0^T \frac{1}{2} \mathbf{u}_c^* \mathbf{J} \, dt\right). \tag{70}$$

The solution $\mathbf{u}_c$ can be expressed in terms of the Green's function of the operator Δ^{-1},

$$\mathbf{u}_c(t) = \int_0^T \mathbf{G}(t,t')\mathbf{J}(t')\,dt' \tag{71}$$

and the final form of the generating function is

$$Z_0[J] = \exp\left(\frac{1}{2}\int_0^T\int_0^T \mathbf{J}^*(t)\mathbf{G}^*(t,t')\,\mathbf{J}(t')\,dtdt'\right). \tag{72}$$

6 - THE CASE OF THE ENSO

[10] proposed a simple model of the ENSO system based on the recharge theory. They showed that ENSO can be described by a simple linear system

$$\begin{aligned}\frac{dh}{dt} &= -rh - \alpha\mu b_0 T\\ \frac{dT}{dt} &= (\gamma\mu b_0 - c)T + \gamma h\end{aligned}$$

where T is the anomaly SST in the West Pacific and h is the anomaly depth of the thermocline in the East. The parameter μ measures the strength of the interaction between the SST and the wind stress. Introducing the vector $\mathbf{q} = (h,\theta)$ we can write it as

$$\frac{d}{dt}\begin{pmatrix} h\\ \theta\end{pmatrix} = \begin{pmatrix} -r & -\alpha\mu b_0\\ \gamma & \gamma\mu b_0 - c\end{pmatrix}\begin{pmatrix} h\\ \theta\end{pmatrix}$$

A coordinate transformation of the vector (h,θ) will allow us to transform the matrix to the standard form

$$\frac{d}{dt}\begin{pmatrix} z_1\\ z_2\end{pmatrix} = \begin{pmatrix} \beta & -w\\ w & \beta\end{pmatrix}\begin{pmatrix} z_1\\ z_2\end{pmatrix} \tag{73}$$

The action for this system is given by

$$S = \frac{1}{2Q}\int_0^T (\dot{\mathbf{z}} - A\mathbf{z})^T(\dot{\mathbf{z}} - A\mathbf{z}) + Tr(A)dt$$

The solution without noise $\mathbf{r}(t)$, around which the action must be expanded, is represented by an exponentially modulated oscillation. The period of the oscillation is w and the time scale of its exponential growth/decay is given by $1/\beta$, the oscillation are damped if $\beta < 0$ and are unbounded in case of $\beta > 0$. Neutral oscillations occur for $\beta = 0$.

$$\mathbf{r}(t) = \begin{pmatrix} \mathrm{e}^{\beta t}\left(z_{10}\cos(tw) - z_{20}\sin(tw)\right)\\ \mathrm{e}^{\beta t}\left(z_{20}\cos(tw) + z_{10}\sin(tw)\right)\end{pmatrix}$$

The solution of the stationarity equation (34) satisfying the boundary conditions that allow the calculation of the fluctuation prefactor is given by the function $\mathbf{g}_c$

$$\begin{pmatrix} \frac{\sinh(\beta t)\left(z_{20}\sin(tw-2Tw)\mathrm{e}^{\beta T}+z_{10}\cos(w(2T-t))\mathrm{e}^{\beta T}-2z_{1T}\cos(w(T-t))+z_{2T}\sin(w(T-t))\right)}{\sinh(\beta T)} \\ -\frac{\sinh(\beta t)\left(2z_{10}\sin(tw-2Tw)\mathrm{e}^{\beta T}-z_{20}\cos(w(2T-t))\mathrm{e}^{\beta T}+z_{2T}\cos(w(T-t))+z_{1T}\sin(w(T-t))\right)}{\sinh(\beta T)} \end{pmatrix}$$

and so the propagator can be written as

$$\begin{aligned} K_0(0,T;z_{10},z_{20},z_{1T},z_{2T}) = {} & \frac{\beta e^{\beta T}}{2\pi Q\sinh(\beta T)} \\ & \exp\left(-\frac{\beta}{2Q\sinh(\beta T)}\left[e^{-\beta T}(z_{10}^2+z_{20}^2)+e^{\beta T}(z_{1T}^2+z_{2T}^2)\right.\right. \\ & \left.\left.+2\sin(wT)(z_{1T}z_{20}-z_{10}z_{2T})-2\cos(wT)(z_{10}z_{1T}+z_{20}z_{2T})\right]\right) \end{aligned} \tag{74}$$

With the choice of parameters proposed in [10], $c=1, \gamma=0.75, r=0.25, \alpha=0.125, b_0=2.5, \mu=2/3$, the system undergoes stable oscillations and the values of the corresponding matrix L are $\beta=0$ and $w=\sqrt{3/32}$. The corresponding propagator can be written as

$$\begin{aligned} K_c = \frac{1}{2\pi QT}\exp\left(-\frac{1}{2QT}\left(z_{10}^2+z_{1T}^2+z_{2T}^2+z_{20}^2\right.\right. \\ \left.\left.-2\cos\left(Tw\right)\left(z_{10}z_{1T}+z_{20}z_{2T}\right)+2\sin(wT)(z_{1T}z_{20}-z_{10}z_{2T})\right)\right) \end{aligned} \tag{75}$$

Fig. 4 shows the probability distribution obtained for a propagator for an initial probability distribution that is a delta function at the origin. It is a gaussian (in the figure only the section for $z_{2T}=0$ is shown) whose standard deviation increases with time. The system is analogous to a Brownian motion with the particle diffusing in the entire space.

The period of the oscillation is close to 20 months and the separate members of the ensemble deviate rapidly as the system evolves. Fig.2 shows the evolution of the individual members of the ensemble as the oscillation gains larger and larger amplitude. The basic linear oscillation is neutral , so it is the stochastic fluctuations that create the amplification effect that results in the flattening of the probability distribution at later times.

For values of μ smaller than the critical value the oscillation is damped, but the stochastic forcing can counterbalance it, slowing the amplification and permitting a statistical equilibrium. Fig. 2 shows the time evolution for the damped case and it is possible to see how the divergence is considerably slowed down. Depending on the magnitude of the stochastic force Q, a different value of μ is necessary for equilibrium.

The probability distribution is correctly estimated by the propagator as it can be seen in Fig. 3. The zeroth order generating function can be obtained from the Green's function as in (71). The Green's matrix can be computed from the fundamental solutions of the system (73) following the results by (Turk). For instance the two point correlation function is given by the second functional derivative of $Z_0(J)$,

$$\langle z_1(x)z_1(t)\rangle = \left(\frac{\delta}{\delta J_1(x)}\frac{\delta}{\delta J_1(t)}Z_0[\mathbf{J}]\right)\Bigg|_{\mathbf{J}=0} =$$
$$\frac{1}{2}\frac{\delta}{\delta J_1(x)}\left(\int_0^T\int_0^T \delta(t-\tau)G_{11}(\tau,\tau')J_1(\tau')d\tau d\tau' + \int_0^T\int_0^T J_1(\tau)G_{11}(\tau,\tau')\delta(t-\tau')d\tau d\tau'\right)Z_0[\mathbf{J}]\Bigg|_{\mathbf{J}=0} =$$
$$\frac{1}{2}\int_0^T \delta(x-\tau')G_{11}(t,\tau')d\tau' + \frac{1}{2}\int_0^T G_{11}(\tau,t)\delta(x-\tau)d\tau = \frac{1}{2}\left(G_{11}(t,x)+G_{11}(x,t)\right) \quad (76)$$

The Green's function G_{11} for the model ENSO problem in the transformed coordinates is given by

$$G_{11}(x,t) = -\frac{Q\,\cos(t\,w-w\,x)\left(\frac{\sinh(\beta t)\,\sinh(\beta\,x)}{\mathrm{e}^{\beta\,T}} - \frac{\sinh(\beta\,T)\,\sinh(\beta\,t)}{\mathrm{e}^{\beta\,x}}\right)}{\beta\,\sinh(\beta\,T)}\,\theta(x-t)$$
$$-\frac{Q\,\cos(tw-wx)\left(\frac{\sinh(\beta\,t)\,\sinh(\beta\,x)}{\mathrm{e}^{\beta\,T}} - \frac{\sinh(\beta T)\,\sinh(\beta\,x)}{\mathrm{e}^{\beta\,t}}\right)}{\beta\,\sinh(\beta\,T)}\,\theta(t-x) \quad (77)$$

and so the standard deviation is given by equal time correlations ($x=t$)

$$\langle z_1(t)z_1(t)\rangle = -\frac{Q\,\cosh(\beta\,T-2\,\beta\,t) - Q\cosh(\beta\,T)}{2\,\beta\,\sinh(\beta\,T)},$$

considering the evolution for a semi-infinite domain as T becomes very large we have

$$\langle z_1(t)z_1(t)\rangle = -\frac{Q\left(\mathrm{e}^{-2\beta t}-1\right)}{2\beta}$$

that has the equilibrium value of

$$\langle z_1(t)z_1(t)\rangle_{eq} = \frac{Q}{2\beta}.$$

It is interesting to note that the same time correlation does not depend on the oscillating part of the solution and the frequency w does not appear anywhere. The autocorrelation for positive lags $\tau = x-t$ is given by

$$\langle z_1(t)z_1(t+\tau)\rangle = \frac{Q\cos(\tau w)\left(1-\mathrm{e}^{-2\beta t}\right)}{2\beta\mathrm{e}^{\beta\tau}}$$

that has the equilibrium value

$$\langle z_1(t)z_1(t+\tau)\rangle_{eq} = \frac{Qe^{-\beta\tau}\cos(\tau w)}{2\beta}.$$

The cross-correlations in these coordinates are identically zero, but going back to the (h, θ) coordinates will recover the correlations shown in [10].

In the same paper a nonlinear extension of the standard model is proposed. The nonlinear terms represent the negative feedback of the thermocline and involve the strength of the coupling between the wind stress and the SST and they are cubic in h and θ. The extra term appears only in the equation for the temperature as

$$-\epsilon(b\theta + h)^3$$

We can use this expression to get the non linear terms in the action (56) to obtain the perturbation expansion in power of the interaction coefficient ϵ that corrects the free (linear) propagator in the presence of nonlinear terms. The expansion is rather tedious and to illustrate the point we will somewhat simplify the system by reducing the nonlinear term to a simple form obtaining a simplified version of the cubic nonlinear term in the system (73) that will result in

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \beta & -w \\ w & \beta \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\varepsilon z_2^3 \end{pmatrix} \tag{78}$$

where ε measures the strength of the nonlinearity and it can be scaled by β.

The action for this system is given by (57) where the $\mathbf{z}$ plays the role of the $\mathbf{g}$. The relevant terms in the action are those deriving from $\boldsymbol{\phi}^T\mathbf{f}(\mathbf{z}+\mathbf{r})$ that in case reduce to the interaction terms between ϕ_2 and z_2, $-\varepsilon\beta\phi_2 g_2^3$. There are also terms deriving from the divergence in the action. In the present case of a cubic interaction those terms are quadratic and in principle they could be included in the explicit linear action, but we will treat them perturbatively to illustrate the point.

The interaction terms are therefore given by:

$$V_I(\phi, z) = \varepsilon\beta\left(\frac{3z_2^2}{2} - \phi_2 z_2^3\right)$$

The generating function for these terms is then given by (64)

$$Z_V(\mathbf{J}) = \exp\left(V_I(\frac{\delta}{\delta\mathbf{J}})\right) Z_0(\mathbf{J})$$

that can be expanded in power of ε,

$$Z_V(\mathbf{J}) = \left[1 + \int_0^T V_I\left(\frac{\delta}{\delta j_1(\tau)}, \frac{\delta}{\delta j_2(\tau)}, \frac{\delta}{\delta k_3(\tau)}, \frac{\delta}{\delta k_4(\tau)}\right) dt + \cdots\right] Z_0(\mathbf{J})\big|_{\mathbf{J}=0}$$

where we have introduced for convenience the numbering $\mathbf{j} = (j_1, j_2)$ and $\mathbf{k} = (k_3, k_4)$. The functional derivatives have to be evaluated at the same time point, τ, and they correspond to the powers of the dynamical variables. The quadratic term corresponds to two functional derivatives with respect to $j_2(t)$ as in (76). The basic rules of functional derivation are given by

$$\frac{\delta f(t)}{\delta g(\tau)} = 0 \qquad \frac{\delta f(t)}{\delta f(\tau)} = \delta(t-\tau)$$

and so the two derivatives in the first term will eliminate all terms with less than two j, k whereas the terms with a larger number of (j,k) will be eliminated by the evaluation at $\mathbf{J} = (\mathbf{j}, \mathbf{k}) = 0$. Due to these mechanisms the derivative select only quadratic terms in the expansion of Z resulting in

$$\frac{3}{2}\int_0^T G_{22}(\tau,\tau)d\tau.$$

The other term will be obtained by taking four derivatives, three with respect to j_2 and one with respect to k_4. There are two such terms

$$j_2 G_{24} k_4 j_2 G_{22} j_2, \quad k_4 G_{42} j_2 j_2 G_{22} j_2$$

using again the method described before the contribution of the $\phi_2 z_2^3$ term is

$$\frac{4!}{8}\int_0^T (G_{24}(\tau,\tau) + G_{42}(\tau,\tau))G_{22}(\tau,\tau)d\tau$$

As an example we will compute the correction of the temporal covariance of z_1 to demonstrate the approach. This covariance is given by the 2-point correlation function

$$\langle z_1(t_1) z_1(t_2)\rangle = \left.\left(\frac{1}{Z_V[\mathbf{J}]}\frac{\delta}{\delta J_1(t_1)}\frac{\delta}{\delta J_1(t_2)} Z_V[\mathbf{J}]\right)\right|_{\mathbf{J}=0}$$

the denominator is given by the expression computed previously

$$1 + \varepsilon\left(\frac{3}{2}\int_0^T G_{22}(\tau,\tau)d\tau + \frac{4!}{8}\int_0^T (G_{24}(\tau,\tau) + G_{42}(\tau,\tau))G_{22}(\tau,\tau)d\tau\right),$$

the numerator is more complicated because now there are two more derivatives. The same arguments used before now lead to the conclusion that only terms with three Green's functions will survive. The problem is combinatorial and it is well known in quantum field theory, it is essentially the same as finding all possible combinations of six points in time: the "external" points, t_1, t_2, and the "internal" points τ that are going to be integrated over. Depending on which of the six j or k the derivatives will be acting the external points will generate different kinds of integrals. The zero order in ε is simply $G_{22}(t_1, t_2)$, but for the first order we need to count the contribution from V_I. The quadratic term in z_2 will result in

$$M_1 \frac{1}{8}\frac{3}{2}\int_0^T G_{22}(t_1,\tau)G_{22}(\tau,t_2)d\tau \tag{79}$$

$$M_2 \frac{1}{8}\frac{3}{2}\int_0^T G_{22}(t_1,t_2)G_{22}(\tau,\tau)d\tau. \tag{80}$$

The combinatorial analysis indicates that in all there are 4 $4!$ terms given by the four time points we are treating (t_1, t_2, τ, τ), organized in such a way that $M_1 = 16$ and $M_2 = 8$. More complicated expressions are obtained from the quartic terms. In this case there are three Green's functions involved G_{22}, G_{24} and G_{42}. Considering first the combination with G_{24} we can see that there are $5! * 3 = 360$ terms,

$$M_3 \frac{1}{3!}\frac{1}{8}\int_0^T G_{22}(t_1,t_2)G_{22}(\tau,\tau)G_{24}(\tau,\tau)d\tau \tag{81}$$

$$M_4 \frac{1}{3!}\frac{1}{8}\int_0^T G_{22}(t_1,\tau)G_{24}(\tau,\tau)G_{22}(\tau,t_2)d\tau \tag{82}$$

$$M_5 \frac{1}{3!}\frac{1}{8}\int_0^T G_{22}(t_1,\tau)G_{22}(\tau,\tau)G_{24}(\tau,t_2)d\tau. \tag{83}$$

with $M_3 = 144$, $M_4 = 144$, $M_5 = 72$. Another 360 terms will come from the symmetric terms containing G_{42}.

However some simplification can be obtained because we can factor the numerator to the first order in ε in such a way as to cancel completely the normalization at the denominator. We can collect the $G_{22}(t_1, t_2)$ to obtain for the numerator,

$$G_{22}(t_1,t_2)\left(1+\varepsilon\int_0^T \frac{3}{2}G_{22}(\tau,\tau)d\tau + 3\varepsilon\int_0^T G_{22}(\tau,\tau)G_{24}(\tau,\tau)d\tau + 3\varepsilon\int_0^T G_{22}(\tau,\tau)G_{42}(\tau,\tau)d\tau\right) + other \quad terms \quad in \quad \varepsilon \tag{84}$$

or at the first order in ε

$$\left(1+\varepsilon\int_0^T \frac{3}{2}G_{22}(\tau,\tau)d\tau + 3\varepsilon\int_0^T G_{22}(\tau,\tau)G_{24}(\tau,\tau)d\tau + 3\varepsilon\int_0^T G_{22}(\tau,\tau)G_{42}(\tau,\tau)d\tau\right)\times (G_{22}(t_1,t_2) + other \quad terms \quad in \quad \varepsilon) \tag{85}$$

first parenthesis cancels with the numerator and we obtain he final expression for the variance

$$\langle z_1(t_1)z_1(t_2)\rangle = G_{22}(t_1,t_2) + other \quad terms \quad in \quad \varepsilon$$

namely, it is the unperturbed variance corrected by the nonlinear terms.

The terms in the perturbation expansion can be expressed with a graphical representation via Feynman diagrams. In our problem there are three kind of propagators, corresponding to the matrix elements of the Green's matrix. The diagonal element generates the propagator of the state variable z and the off diagonal terms, that turn out to be symmetric, generate the propagator connecting the state variable to the auxiliary variables ϕ. We can graphically express the Green's function $G_{22}(t_1, t_2)$ with a straight line. The G_{24} propagator instead can be seen as a dashed-continuous line. The points t_1 and t_2 are the external lines of the graph, the time point τ is recurring twice and therefore is special, because it has two lines that must be connected with the other point

The quadratic terms (80) can be written graphically as in Fig. 7. The (b) graph in the figure represents the integral where we can factor out the $G_{22}(t_1, t_2)$ propagator. It is an example of the fact that these kinds of terms show up graphically as made up of separate parts, the so-called "disconnected" graph, in this example it is the product of $G_{22}(t_1, t_2)$ and $\int_0^T G_{22}(\tau, \tau) d\tau$.

The terms corresponding to $z_2^3 \phi_2$ are more complicated. The internal vertex is of order four and it has four lines, that must be connected with two external points. A four line vertex corresponds to the product of two Green's functions, in this case a G_{22} and a G_{24}, because there are only two external lines the other two lines must be closing on themselves. The graphs are shown in Fig.8, without showing all the possible symmetries and exchanges that go into producing all the 720 terms.

The disconnected graphs are the product of the component graphs, so the final correction to the variance or 2-point correlation can be written in the form

$$\langle z_1(t_1) z_1(t_2) \rangle = G_{22}(t_1, t_2) + M_2 \frac{\varepsilon}{8} \frac{3}{2} \int_0^T G_{22}(t_1, t_2) G_{22}(\tau, \tau) d\tau +$$
$$\varepsilon M_4 \frac{1}{3!} \frac{1}{8} \int_0^T G_{22}(t_1, \tau) G_{24}(\tau, \tau) G_{22}(\tau, t_2) d\tau + \varepsilon M_5 \frac{1}{3!} \frac{1}{8} \int_0^T G_{22}(t_1, \tau) G_{22}(\tau, \tau) G_{24}(\tau, t_2) d\tau \quad (86)$$

The results are shown in Fig.9. The figure shows the time evolution of the variance at equal times of an ensemble of 2000 numerical simulations. The solid line for the linear case concurs with the theoretical value at equilibrium, $Q/2\beta = 8$, within the errors. The first order estimate of the nonlinear equilibration gives 7.35 and 6.50 for $\varepsilon = 0.1$ and $\varepsilon = 0.3$ that are also agreeing with results.

7 - CONCLUSIONS

This paper has shown that the path integral formulation and functional methods can be used for stochastic equations derived from the type of equation of motion that are used to describe the atmosphere and the ocean. In this framework these equations pose special problems as, in general, they do not admit a potential and are only first order in time resulting in an action that has a special form that introduces coupling terms between the velocity terms and the forcing function.

This problem prevents a straightforward application of the method as in quantum physics but it can be treated by a careful consideration of the boundary conditions. It is also shown that problems in dimensions higher than one with no potential can still be treated using the Stratonovich-Hubbard transformation. A perturbation expansion can then be designed for nonlinear cases based on the calculation of the generating function for the n-points correlation functions and Feynman diagrams can be introduced.

The method is general and most of the methods, concepts and techniques can be extended to stochastic partial differential equations.

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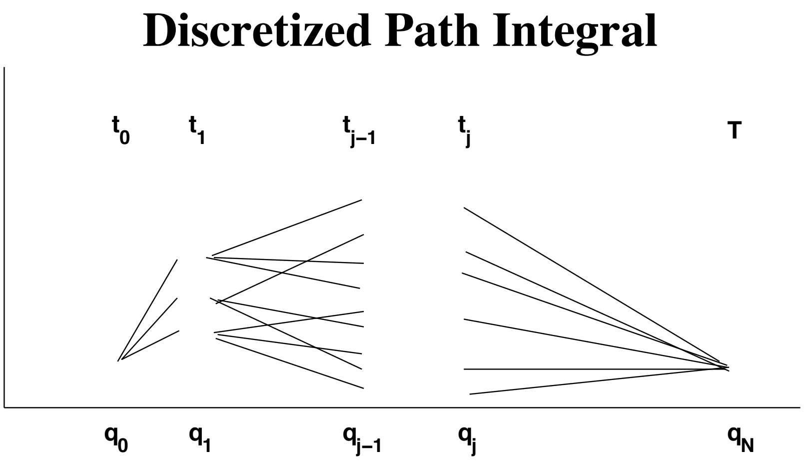
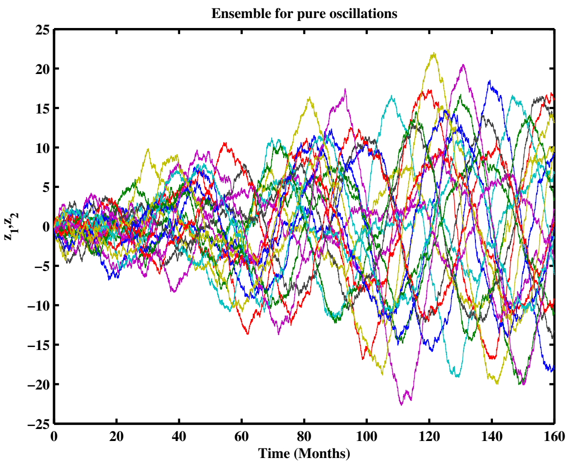
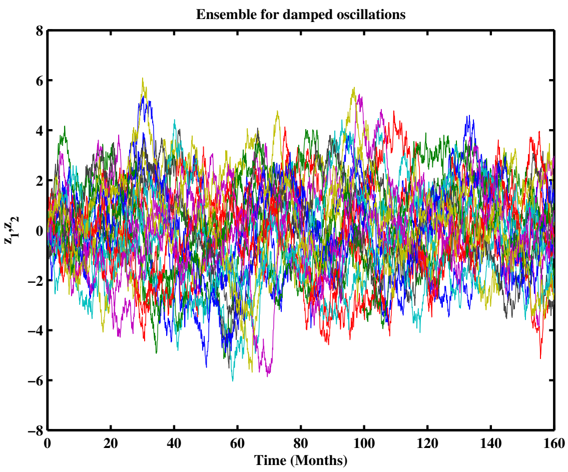


Figure 1: Discretization of the path integral. The initial q_0 and final q_N variable are not integrated over



(a) Evolution for $\mu = 2/3$



(b) Evolution for $\mu = 1/2$

Figure 2: The time evolution of 10 members for the critical case (a) and the subcritical case (b).

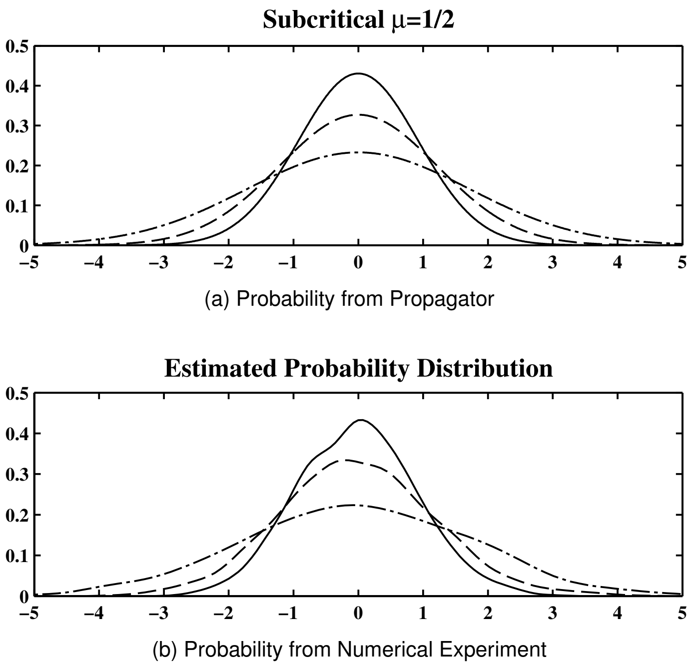
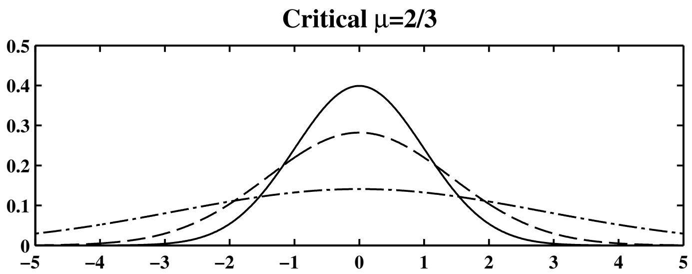
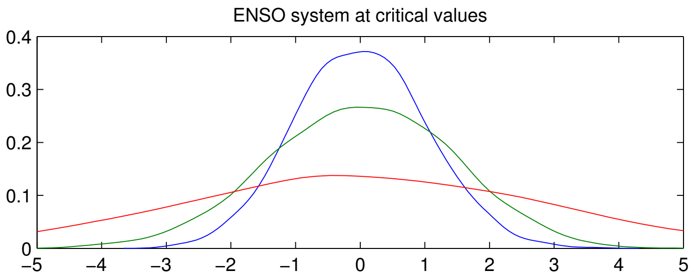


Figure 3: The probability distribution for the subcritical case ($\mu = 1/2$) from the propagator (a) and from 2000 numerical experiment (b). The solid line correspond to T=1, the dashed line to T=2 and the dot-dashed line to T=8.



(a) Propagator



(b) Numerical Experiment

Figure 4: The section of the probability distribution at $z_{2T} = 0$ for a particle at different final times starting at the origin obtained from the theoretical propagator (a); probability distribution for z_{2T} obtain from numerical experiments of the linear system with stochastic forcing with a sample of 2000 integrations (b) at the same final times as above.

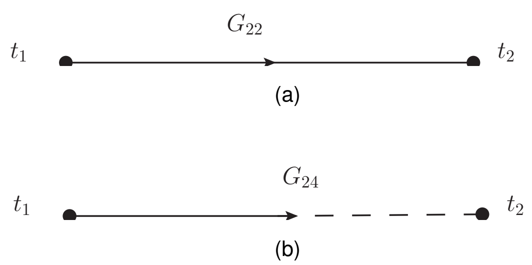


Figure 5: The propagators of the system: (a) the propagator for the variables (z_2, z_2) (b) the propagator for the variables (z_2, ϕ_2). A corresponding propagator can be obtained exchanging 2 and 4.

Figure 6: The internal vertex τ. (a) for the quadratic term z_2^2, (b) for the the quartic term $z_2^3\phi_2$

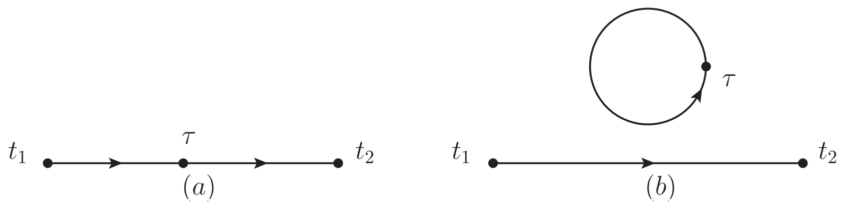


Figure 7: The graphical representation of the propagator for G_{24}

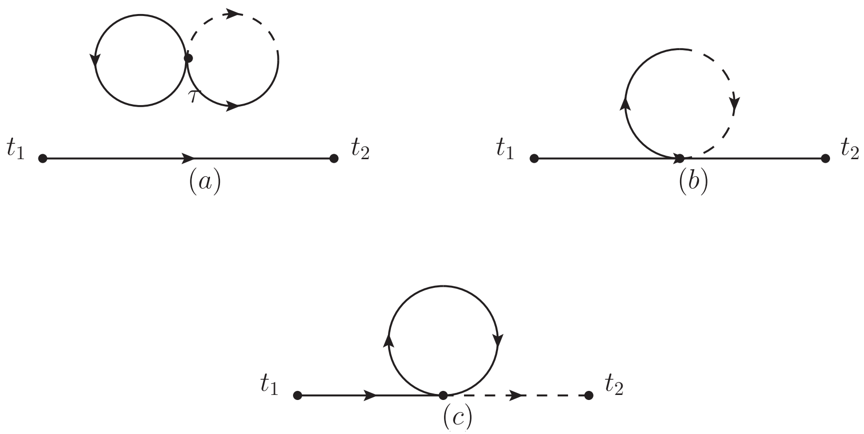


Figure 8: The terms of the perturbation expansion for the 2-point correlation, the variance. The full contribution can be obtained by using symmetry over all the vertices and adding the graphs obtained exchanging 2 with 4: (a) disconnected graph, corresponding to (81), (b) graph with G_{24} integrated over the internal vertex τ, corresponding to (82), (c) graph with G_{24} into an external point corresponding to (83).

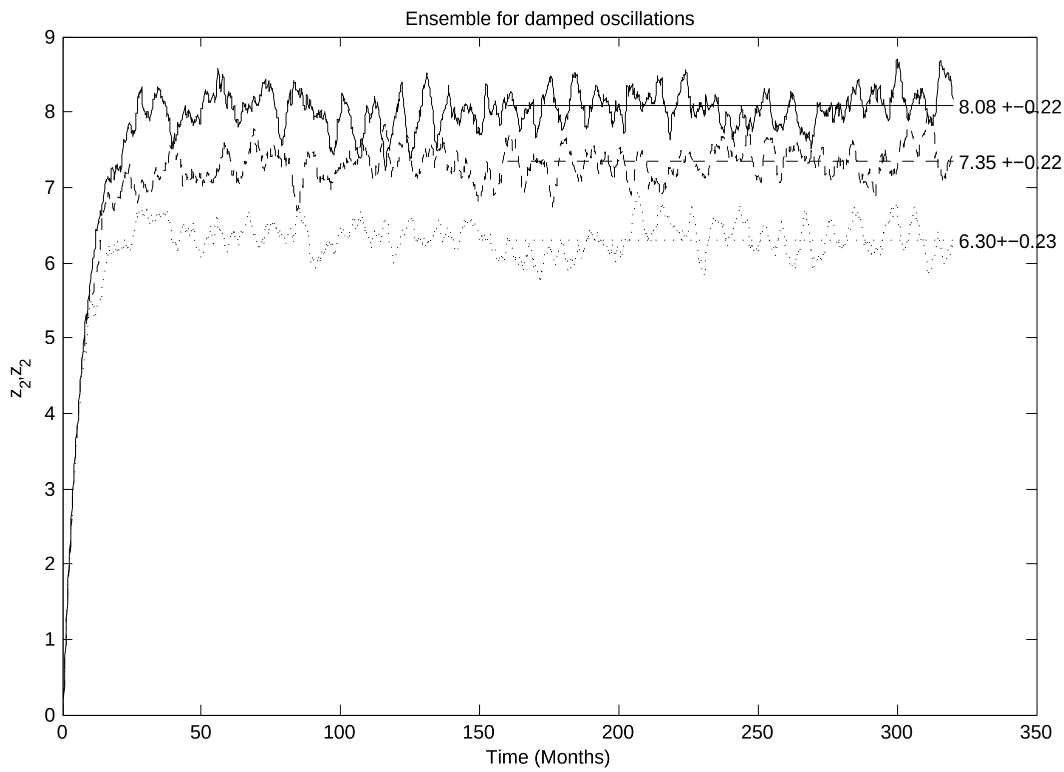


Figure 9: The evolution of the equal time variance for an ensembles of 2000 simulations for the test system. The averaged variance computed after equilibration and its standard deviation is shown on the right. The solid line represent the linear system, the dashed line is the nonlinear system with $\varepsilon = 0.01$ and the dotted line is the nonlinear system with $\varepsilon = 0.03$

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