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Information Sharing Networks in Linear Quadratic Games

By **Sergio Currarini**
University of Leicester
Università Ca' Foscari di Venezia
Fondazione Eni Enrico Mattei (FEEM)
Centro Euro-Mediterraneo sui Cambiamenti Climatici (CMCC)
sc526@le.ac.uk

Francesco Feri
Royal Holloway, University of London.

SUMMARY We study the bilateral exchange of information in the context of linear quadratic games. An information structure is here represented by a non directed network, whose nodes are agents and whose links represent sharing agreements. We first study the equilibrium use of information in any given sharing network, finding that the extent to which a piece of information is "public" affects the equilibrium use of it, in line with previous results in the literature. We then study the incentives to share information ex-ante, highlighting the role of the elasticity of payoffs to the equilibrium volatility of one's own strategy and of one's opponents' strategies. For the case of uncorrelated signals we fully characterize pairwise stable networks for the general linear quadratic game. For the case of correlated signals, we study pairwise stable networks for three specific linear quadratic games - Cournot oligopoly, Keynes' beauty contest and Public good provision - in which strategies are substitute, complement and orthogonal, respectively. We show that signals' correlation favors the transmission of information, but may also prevent all information form being transmitted.

Keywords: Information Sharing, Networks, Bayesian equilibrium, Beauty Contest, Oligopoly.

JEL: D43, D82, D85, L13

INTRODUCTION

Linear quadratic games have played a key role in the analysis of games of incomplete information in economics. The implied linear best reply functions allow for the existence of a Bayesian equilbrium in affine strategies (see Radner 1962, Angelitos and Pavan, 2007). Specific examples include linear Cournot oligopoly, Keynes' Beauty Contest and public good games with linear benefits and quadratic costs. The analytical tractability of linear quadratic games has motivated an extensive theoretical effort to understand the use of information in environment with fundamental uncertainty, tracing such use to strategic features of the game such as complementarity and substitutability, and drawing welfare and policy implications (see Morris and Shin, 2002 and Angelitos and Pavan, 2007). Recently, linear quadratic games have been shown to allow for a precise characterization of equilibrium behavior when the patterns of strategic interaction between players is represented by a network, in which each pair of agents interact with a given intensity and sign (see Ballester et al. 2006).

One issue that is strictly related to the equilibrium used of private information is the possibility that agents share their private information before engaging in non cooperative behavior. In the framework of imperfect market competition, this issue has spurred an extensive debate that dates back to the seminal contributions of Novshek and Sonnenschein (1982) and Vives (1985). Understanding the incentives of firms to share information before engaging in market competition is important since it can help draw a line between collusive market behavior (which is suboptimal) and pure sharing of information prior to competition (which is socially desirable). One main insight from this body of literature is that incentives to share are associated with either strategic complementarity or weak substitutability, be it induced by products differentiation, by cost convexity or by price competition (see Vives,1985, Kirby, 1988 and Raith, 1996). While certainly of great relevance for policy and for welfare, these conclusions rest on the specifics of the imperfect competition model, and little is known about the incentives to share in other instances of the linear quadratic game, possibly reflecting different incentives and different types of economic interactions, such as the beauty contest or public good games. Moreover, while the traditional approach has mainly studied the incentives of firms to jointly and universally disclose all private information[1] , in many economic contexts agents may agree to share information in smaller groups or in pairs, by means of private agreements of various degrees of commitment.

These consideration motivate the present analysis of information sharing in general linear quadratic games. We approach information sharing from a bilateral perspective, assuming that each pair of agents can commit (ex-ante) to mutually (and truthfully) disclose their own private information to each other. The ex-ante assumption allows us to dismiss all strategic considerations that relate to the inference of other agents' information from their sharing behavior. In this context, an information structure is well represented by a non directed network, in which an agent's private information consists of the signals observed by her "neighbors" in the network. Compared to previous literature where each agent observes a "private" signal (only revealed to her) and a "public" signal (observed by all) (see Morris and Shin, 2002 and Angelitos and Pavan, 2007), here each signal is public to a specific subset of agents - the neighbourhood.

Agents decide whether to engage in sharing agreements prior to observing their own private signal.

[1] With the exception of Kirby (1988) and Malueg and Tsutsui (1996)

Once agreements are made and signals are observed, players play a linear quadratic game of incomplete information, in which information sets are given by the network. We first characterize the equilibrium use of information in the network. We find that the sensitivity of each player's strategy to each observed signal in the network depends on the strategic nature of the underlying game. In line with previous works in the literature, strategic complementarities induce agents to use more intensively those signals that are more public, by this meaning those that are observed and used more intensively by other agents in the network. Opposite conclusions apply to games with strategic substitutes. As we will see, this implies that more public signals are less attractive to acquire when actions are substitutes and more attractive when actions are complements.

In section 4 we study the incentives to share information. Our focus is on "pairwise stable" networks, providing no two agents with the incentive to form a new link, and no agent with the incentive to unilaterally sever an existing link (see Jackson and Wolinsky, 1996). Differently from all previous works on information sharing in oligopolies, our analysis cannot exploit the symmetry of agents equilibrium strategies even when the underlying game and the statistical structure are in all respects symmetric. In fact, the gains accruing to an agent severing a link or to two agents forming a new one are assessed by evaluating the (expected) change in payoffs due to a local change in the existing network. Lacking symmetry, the analysis of incentives becomes soon too complex for a comprehensive characterization of stable networks for all statistical models. Most of the complexity is due to the widespread interrelation of agents' equilibrium use of information, due to the inherent correlation of signals (unconditional and, possibly, conditional).

In section 4.1, as a first tractable benchmark, we study the case of independent signals. This limit case is obtained by setting a conditional correlation which exactly offset the natural signals' correlation induced by the state of the world. This artificial case, first suggested in Raith (1996), generates a model which is formally equivalent to the model of imperfect competition with i.i.d. signals used in Gal-Or (1985), to which some of our results apply. This approach is quite standard in common value problems such as auctions (see Bulow and Klemperer, 2002, Levin, 2004 and Tan, 2012), and has been employed by Hagenbach and Koessler (2010) in their analysis of strategic information transmission in networks (see below for a discussion of the differences between their approach and the present paper). Using this case as a benchmark will also prove useful in section 4.2 to understand the role of correlation in shaping the incentives to share information. For this case we provide a full characterization of pairwise stable networks in the general linear quadratic problem. We find that the incentives to share, and the architecture of stable networks, crucially depend on how sensitive payoffs are to the volatility of one's own action, on aggregate volatility, and to the correlation between opponents' actions and the state of the world. When payoffs mostly depend on one's own equilibrium volatility (as in the Cournot case), stable architectures are made of fully connected components of increasing sizes (possibly including singletons); when strategies are complements, only the complete network survives among such structures. When payoffs are also sensitive to aggregate volatility, then incomplete stable structures may emerge, even when strategies are complements. In particular, we show that regular incomplete networks can be stable when aggregate volatility is detrimental to one's own payoff. This is somewhat contrary to the common perception that strategic complementarities should provide agents to share all of their available information. This perception is correct in the specific case of Keynes' Beauty Contest, where the effect of aggregate volatility is bounded in magnitude, but not in more general frameworks. Finally, we study the case in which payoffs are also sensitive to the correlation of opponents strategies and the state of the world. Here we focus on games in which strategies are orthogonal, and focus on the interplay of the various parameters on incentives to share and stability. We fully characterize pairwise stable networks, show that such networks can have incomplete architecture, and may fail to exist for certain ranges of parameters.

In Section 4.2 we finally turn to the case of correlated signals. Here, we focus on three specific cases of the linear quadratic game, each stressing the role of one of the components of the general model. For the Cournot game, where only one's own equilibrium volatility is payoff relevant, we show that two firms may find it profitable to share information when no other firms do. In particular, incentives to share build up when signals are conditionally correlated, and one extra signals provides better inference on the other, unobserved, signals. This "global" improved inference comes without the disclosure of one's own signal to any other firms except for the new sharing partner. When signals' correlation is not too weak, these incentives are sufficient to rule out the pairwise stability of the empty network, for any level of products' differentiation. Moreover, the complete network is pairwise stable for all levels of signals' correlation. We further study how signals' correlation affects the incentives to share information in the contest of a four-firm example with perfect substitutes, for which we fully characterize pairwise stable networks as a function of signals' correlation. We find that while correlation creates additional incentive to share information, such incentives decrease with the number of observed signals, and may disappear before all information is shared. Finally, we show that in the Beauty Contest the complete network is always pairwise stable, while the empty network never is. For the public good game we show that the complete is the unique pairwise stable information structure.

We finish by commenting on some recent literature on information transmission on networks. Galeotti, Ghiglino and Squintani (2009) study the "many sender - many receivers" game of cheap talk, interpreting the flows of truthful information as directed links in a network. The focus is there on the incentives to truthfully report the observed information, and while in their model all agents would benefit from the disclosure of all available information, this may not be feasible in equilibrium. Hagenbach and Koessler (2010) enrich this basic cheap talk model by adding a coordination motive as in Keynes' Beauty Contest of our section 4.1. They keep the analysis tractable by assuming that the state of the world takes the form of the sum of agents' independent signals, essentially ruling out correlation as we do in section 4.1. Our proposition 5 (showing that the complete network is the unique pairwise stable structure in the Beauty Contest) can indeed be viewed as a corollary of their proposition 2, where it is shown that agents always benefit from disclosing or receiving additional information. Our proposition 8 (showing that the complete network is a pairwise stable structure), instead, extends the analysis of the Beauty Contest to correlated signals, while the rest of our analysis, where we study other classes of linear quadratic games, is less related to theirs and closer in spirit to the quoted literature on oligopolies. As a general comment, our focus is on the incentives to share information at the ex-ante stage, as they result from the gains from acquiring and the possible losses from disclosing. Since we assume identical preferences, truth-telling is not an issue in the Beauty contest, while there is no truth-telling equilibrium in the Cournot game (as shown in Zen).

The paper is organized as follows: section 2 presents the economic and statistical models. Section 3 studies how agents use the available information in the network. Section 4 studies information sharing with and without correlation of signals. Section 5 concludes the paper.

THE MODEL

We consider a set of n agents, each agent i choosing an action $a_i \in \Re$. Agent $i's$ utility is given as a function of her action a_i, the sum of other agents' actions $A_i = \sum_{j \neq i} a_j$ and a parameter θ denoted as the "state of the world":

$$u_i(a_i, A_{-i}, \theta) = \begin{pmatrix} \lambda_a \\ \lambda_A \\ \lambda_\theta \end{pmatrix}' \begin{pmatrix} a_i \\ A_i \\ \theta \end{pmatrix} + \begin{pmatrix} a_i \\ A_i \\ \theta \end{pmatrix}' \begin{pmatrix} \gamma_a & \gamma_{aA} & \gamma_{a\theta} \\ \gamma_{aA} & \gamma_A & \gamma_{A\theta} \\ \gamma_{a\theta} & \gamma_{A\theta} & \gamma_\theta \end{pmatrix} \begin{pmatrix} a_i \\ A_i \\ \theta \end{pmatrix} \tag{1}$$

where the γ coefficients in the interaction matrix measure the quadratic relations between agents' utilities, agents' actions and the state of the world, while the λ coefficient measure the linear relations between agents' utilities, agents' actions and the state of the world.

The state of the world θ is assumed to be a random variable of the form

$$\theta = \mu_\theta + \epsilon$$

where $\epsilon \sim N(0, t)$ and μ_θ is a constant and its value is common knowledge. Agents' information is structured as follows. Each agent i receives a private noisy signal y_i about ϵ, with

$$y_i = \epsilon + \eta_i$$

where $\eta_i \sim N(0, u)$ for all i, and where $cov(\eta_i, \eta_j) = u_n$ for all i, j. We also assume that $cov(\eta_i, \epsilon) = 0$ for all i. For notational convenience, we will denote by $p_s = (t + u)$ the variance of signals and by $p_n = (t + u_n)$ the covariance of signals.

We will consider the possibility that agents share their information by means of bilateral and truthful sharing agreements; this means that agent i is allowed to observe agent j's signal if and only if he reveals his own signal to agent j. Sharing agreements may be non transitive, in the sense that information sharing between agents i and j and between agents j and k need not imply information sharing between agents i and k. Formally, the information structure induced by such agreements is represented by a non directed *network* g, in which each link ij denotes a bilateral agreement between agents i and j. We denote by $N_i(g) \equiv \{j : ij \in g\} \cup \{i\}$ the set of neighbours of i in g (including i) and we denote by $n_i^g = |N_i(g)|$ the number of such neighbors. In other words, n_i^g is the number of signals observed by i in g. The information available to agent i in the information structure g is therefore $I_i(g) \equiv \{y_j : j \in N_i(g)\}$, that is the set of signals observed by the neighbors of i. We will use the notation $g + ij$ to denote the network obtained by adding to g the link $ij \notin g$, and $g - ij$ to denote the network obtained by severing the link $ij \in g$ from g.

This model of linear quadratic utility and information sharing has many application in different fields of economic analysis. Here a three such examples.

Beauty Contest (Morris and Shin (2002), Howerbach and Kossner (2010)). There are n agents, each setting an action a_i. Each agent i suffers a loss which increases quadratically in the distance between her action and the average action chosen by the opponents, and a loss which increases quadratically with the distance between a_i chosen action and the realization of a random state of the world θ:

$$u_i(a_i, A_i, \theta) = -v(a_i - \theta)^2 - (1 - v)(a_i - \frac{A_i}{n-1})^2.$$

Using the notation of the present paper, we have: $\gamma_a = -1$; $\gamma_\theta = -v$; $\gamma_{a\theta} = 2v$; $\gamma_A = -\frac{(1-v)}{(n-1)^2}$; $\gamma_{aA} = \frac{2(1-v)}{n-1}$. All other coefficients are zero.

Cournot Oligopoly (Vives, 1985, Kirby, 1988, Raith, 1996). There are n agents competing in a common market with inverse demand function:

$$p = \theta - a_i - \epsilon A_i$$

where a_i denotes agent i's output, A_i denotes the aggregate output of i's competitor, and ϵ captures product differentiation. Firms produce with no costs. Using the notation of the present paper, we have: $\gamma_a = -1$; $\gamma_{a\theta} = 1$; $\gamma_{aA} = -\epsilon$. All other coefficients are zero.

Public goods (Ray and Vohra, 1999). Each agent i contributes the amount a_i to a global public good, and has utility function:

$$u_i(A_i, a_i) = \theta(A_i + a_i) - ba_i^2.$$

The actual value of the public good in a random variable θ about which each agent receives a noisy signal. evaluations A_i of the public good. Using the notation of the present paper, we have: $\gamma_a = -b$; $\gamma_{A\theta} = 1$; $\gamma_{a\theta} = 1$. All other coefficients are zero.

USE OF INFORMATION IN THE NETWORK

With each possible information structure g we associate the Bayesian Nash Equilibrium of the game in which each agent i sets her action a_i in order to maximize her expected payoff (see (1)), given the available information - determined by i's links in g - and given the optimal decisions of the other agents. Formally, a Bayesian Nash equilibrium associated with g is a family of functions a_i^g mapping, for each $i \in N$, the available information $I_i(g)$ into a choice $a_i^g(I_i(g))$, and solving for each agent i the following problem:

$$a_i^g(I_i(g)) = \arg\max_{a_i \in \Re} E\left[u_i\left(a_i, A_i^g(I_{-i}(g))\right), \theta | I_i(g)\right], \tag{2}$$

where we have denoted by $A_i^g(I_{-i}(g))$ the sum of strategies of all agents but i. Note that the terms γ_A, $\gamma_{A\theta}$, λ_A, λ_θ, γ_θ do not affect the equilibrium strategies; they however affect welfare, and will be therefore relevant in determining agents' incentives to form links.

The optimal choice of agent i as a function of i's information is obtained maximizing (1) with respect to a_i. The first order condition is:

$$a_i^g\left(I_i(g)\right) = -\frac{\lambda_a + \gamma_{a\theta} E\left[\theta | I_i(g)\right] + \gamma_{aA} E\left[A_i^g | I_i(g)\right]}{2\gamma_a} \tag{3}$$

As in Angeletos and Pavan (2007), we will assume $\gamma_a < 0$ and $\gamma_a + (n-1)\gamma_{aA} < 0$. Standard results (see Radner, 1962 and Angeletos and Pavan, 2007) can be used to establish the existence of a unique Bayesian Nash Equilibrium for all information structures g, with the equilibrium strategies affine in the observed signals, i.e.:

$$a_i^g\left(I_i(g)\right) = \alpha_i^g + \sum_{y_j \in I_i(g)} \beta_{ij}^g y_j, \qquad i = 1, 2, ...n. \tag{4}$$

The following proposition derives the system defining the equilibrium α_i^g and β_{ij}^g coefficients in the Bayesian game with information structure g.

Proposition 1 *The Bayesian Nash equilibrium of the game with payoff functions (1) and information structure described by the network g is characterized by the following system:*

$$\alpha_i^g = -\frac{1}{2\gamma_a}\left(\lambda_a + \gamma_{a\theta}\mu_\theta + \gamma_{aA}\sum_{j\neq i}\alpha_j^g\right) \tag{5}$$

$$\beta_{ih}^g = -\frac{1}{2\gamma_a}\left(\gamma_{a\theta}k_1^{ig} + \gamma_{aA}\left(\sum_{j\in N_h\setminus i}\beta_{jh}^g + \sum_{z\notin N_i}\sum_{j\in N_z}k_2^{ig}\beta_{jz}^g\right)\right), \ \forall h \in N_i; \tag{6}$$

where

$$k_1^{ig} = \frac{t}{p_s + \left(n_i^g - 1\right)p_n};\ k_2^{ig} = \frac{p_n}{p_s + \left(n_i^g - 1\right)p_n}$$

are the updating coefficients that agent i applies to each $y_j \in I(g_i)$ to take the following expectation on the state θ and of the signals y_h, for $h \notin N_i(g)$, respectively:

$$E\left[\theta | I_i(g)\right] = \mu_\theta + k_1^{ig}\sum_{j\in N_i} y_j; \tag{7}$$

$$E\left[y_h | I_i(g)\right] = k_2^{ig}\sum_{j\in N_i} y_j. \tag{8}$$

From proposition 1 we obtain a first insight in how the information structure g affects the way in which agents use their available information. First, from (5) it is directly verifiable that coefficients α_i^g are the same in all networks and for all agents, which allows to denote $\alpha_i^g = \alpha$.

Condition (6) describes the way in which information is used in equilibrium as a function of the whole network. To fix ideas, assume that $\gamma_{a\theta} > 0$, so that agents choices move together with the state of the world. The coefficient that i applies to signal $y_h \in I_i(g)$ is equal to the sum of the term $(-\frac{\gamma_{a\theta}}{2\gamma_a}k_1^{ig})$ and of the two summations in the second bracket. Both summations amplify the effect of the first term if actions are strategic complements ($\gamma_{aA} > 0$), and weakens the effect of the first term if strategies are strategic substitutes ($\gamma_{aA} < 0$). Both summations measure the reactions of i's opponents that are correlated to signal $y_h \in I_i(g)$; the first refers to the reaction of the opponents that observe $y_h \in I_i(g)$, the second to the opponents' reactions to signals that agent i does not observe, but that are correlated to signal $y_h \in I_i(g)$. Both terms tend to amplify the use of signal $y_h \in I_i(g)$ by agent i when there is an incentive to correlate with other agents (complements), and to reduce the use of signal $y_h \in I_i(g)$ when the incentive is to diversify from the other agents (substitutes). In the terms used by previous works in the literature (e.g., Morris and Shin, 2002), these summations refer to how "public" the signal y_h is in the system, that is, how much that signal is used by other agents to set equilibrium actions. Formally, conditions (5)-(6) directly imply the following proposition.

Proposition 2 *For all $ih \in g$ and $ij \in g$:*

$$\beta^g_{ih} - \beta^g_{ij} = \frac{\gamma_{aA}}{2\gamma_a}\left(\sum_{k\in N_j\setminus i}\beta^g_{kj} - \sum_{k\in N_h\setminus i}\beta^g_{kh}\right). \tag{9}$$

Proposition 2 describes the following equilibrium effect: under strategic substitutes, agents react more to those signals to which, in aggregate, other agents react less, while the opposite holds under strategic complements. The next example illustrates equilibrium use of information in a network where agents' position present stark differences - the star.

Example 1 *We consider the star network g^s with 4 agents, where the central agent i receives a signal which is observed by all agents, and each other signal is observed by the receiving agent h and by the central agent. Equilibrium coefficients solve the following system of equations:*

$$\beta^{g^s}_{ii} = -\frac{1}{2\gamma_a}\left(\gamma_{A\theta}k_1^{ig} + \gamma_{aA}3\beta^{g^s}_{ih}\right) \tag{10}$$

$$\beta^{g^s}_{ih} = -\frac{1}{2\gamma_a}\left(\gamma_{A\theta}k_1^{ig} + \gamma_{aA}\beta^{g^s}_{hh}\right) \tag{11}$$

$$\beta^{g^s}_{hh} = -\frac{1}{2\gamma_a}\left(\gamma_{A\theta}k_1^{hg} + \gamma_{aA}(\beta^{g^s}_{ih}(1+2k_2^{hg}) + 2k_2^{hg}\beta^{g^s}_{hh})\right) \tag{12}$$

$$\beta^{g^s}_{hi} = -\frac{1}{2\gamma_a}\left(\gamma_{A\theta}k_1^{hg} + \gamma_{aA}(\beta^{g^s}_{ii} + \beta^{g^s}_{ih}2k_2^{hg} + 2\beta^{g^s}_{hi} + 2k_2^{hg}\beta^{g^s}_{hh})\right) \tag{13}$$

where we have used symmetry where possible. We obtain the following coefficients for cases of strategic complements ($\gamma_{aA} = .1$) and substitutes ($\gamma_{aA} = -1$), and for different levels of signals' correlation:

	$p_n = .6\,;\gamma_{aA} = -1$	$p_n = .8\,;\gamma_{aA} = -1$	$p_n = .6\,;\gamma_{aA} = .1$	$p_n = .8\,;\gamma_{aA} = .1$
β^g_{ii}	.014	.009	.117	.099
β^g_{ih}	.047	.037	.097	.081
β^g_{hh}	.083	.071	.171	.153
β^g_{hi}	.050	.042	.191	.171

We see that signals which are more public (y_i) are used less intensively than more private ones (y_h) under strategic substitutes, and more intensively under complements. Moreover, periphery agents, who are endowed with fewer pieces of information, use information more intensively. Signals' correlation has a negative impact under both substitutes (as expected) and complements.

INFORMATION SHARING

We study the incentives to share information at the ex-ante stage. For each network g, we denote by $u_i^e(g)$ the ex-ante expected utility for agent i, given that g describes the information structure of the Bayesian game played at the interim stage. The utility $u_i^e(g)$ is obtained by taking the expectation of the interim utility $E\left[u_i|I_i(g)\right]$ over all possible realizations of i's information $I_i(g)$. The interim utility is given by:

$$\begin{aligned} E\left[u_i|I_i(g)\right] = \lambda_a a_i^g(I_i(g)) + \lambda_A E\left[A_i^g|I_i(g)\right] + \lambda_\theta E\left[\theta|I_i(g)\right] + \gamma_a a_i^g(I_i(g))^2 + \\ +\gamma_A E\left[(A_i^g)^2|I_i(g)\right] + \gamma_\theta E\left[\theta^2|I_i(g)\right] + \gamma_{aA} a_i^g E\left[A_i|I_i(g)\right] + \\ +\gamma_{a\theta} a_i^g(I_i(g)) E\left[\theta|I_i(g)\right] + \gamma_{A\theta} E\left[A_i^g\theta|I_i(g)\right]. \end{aligned} \tag{14}$$

Together with the first order condition (3), (14) yields the following expression:

$$\begin{aligned} E\left[u_i|I_i(g)\right] = -\gamma_a \cdot a_i^g(I_i(g))^2 + \lambda_A \cdot E\left[A_i^g|I_i(g)\right] + \lambda_\theta E\left[\theta|I_i(g)\right] + \\ +\gamma_A \cdot E\left[(A_i^g)^2|I_i(g)\right] + \gamma_\theta \cdot E\left[\theta^2|I_i(g)\right] + \gamma_{A\theta} \cdot E\left[A_i^g\theta|I_i(g)\right]. \end{aligned} \tag{15}$$

Note now that, given the linear specification of equilibrium strategies in (4), we can express the variances and covariance of equilibrium strategies as follows:

$$var(a_i^g) = \sum_{h\in N_i^g} (\beta_{ih}^g)^2 p_s + 2\sum_{h\in N_i}\sum_{k<h} \beta_{ih}^g \beta_{ik}^g p_n; \tag{16}$$

$$var(A_i^g) = p_s \cdot \left[\sum_j (B_{ij}^g)^2\right] + p_n \cdot \left[2\cdot \sum_j \sum_{k<j} B_{ij}^g \cdot B_{ik}^g\right]; \tag{17}$$

$$cov\left(A_i^g, \theta\right) = t \cdot \sum_j B_{ij}^g. \tag{18}$$

where $B_{ij}^g \equiv \sum_{h\in N_j \setminus i} \beta_{hj}^g$ denotes the aggregate reaction to signal j by i's opponents.

We now use the above expressions to derive the ex-ante equilibrium utility in any given network g:

$$\begin{aligned} u_i^e(g) = \lambda_A \cdot (n-1)\cdot \alpha + \alpha^2 \cdot \left(\gamma_A \cdot (n-1)^2 - \gamma_a\right) + (\lambda_\theta + \gamma_\theta \mu_\theta)\,\mu_\theta + \gamma_{A\theta}\cdot(n-1)\cdot\alpha\cdot\mu_\theta \\ + \gamma_\theta \cdot var\left(\theta\right) + \gamma_A \cdot var\left(A_i^g\right) + \gamma_{A\theta}\cdot cov\left(A_i^g,\theta\right) - \gamma_a \cdot var\left(a_i^g\right). \end{aligned} \tag{19}$$

We can now express the difference $[u_i^e(g') - u_i^e(g)]$ in agent i's expected utility in g and g'. This difference is measured in (20) as the sum of three terms, expressing the change, when passing from g to g', in the variance of other agents actions, of i's action and in the covariance between other agents' actions and the state of the world:

$$\gamma_A \cdot \left[var\left(A_i^{g'}\right) - var\left(A_i^g\right)\right] + \gamma_{A\theta} \cdot \left[cov\left(A_i^{g'}, \theta\right) - cov\left(A_i^g, \theta\right)\right] - \gamma_a \cdot \left[var\left(a_i^{g'}\right) - var\left(a_i^g\right)\right] \quad (20)$$

Inspection of condition (20) provides insights in the sources of the incentives that a generic agent i has to induce a given network g' from a network g. The first term measures the effect of the change in the variance of the aggregate actions of other agents, keeping all other things equal. This effect is measured by the parameter γ_A, the coefficient that measures the effect on utility of the square of other agents' actions. This term is null in public good games and Cournot games, and is negative in beauty contest games with complements, where larger differences between one's own action and average opponents' actions are weighted more due to the quadratic loss function. The second term measures the effect of a change in the covariance of other agents' actions and the state of the world: this effect is measured by the coefficient controlling for the interaction effect of i's opponents actions and the state of the world, and is non null only in public good example. The last term measures the incentives coming from a change in agent i's variance in equilibrium, and is measured by the own quadratic coefficient $\gamma_a < 0$, implying that increased variance of one's own action is desirable in all linear quadratic games.

We will study the structure of *pairwise stable* networks (see Jackson and Wolinsky (1996)), in which no pair of agents has an incentive to form a new link and no agent has an incentive to unilaterally sever an existing link:

Definition 1 *The network g is* **pairwise stable** *at the ex-ante stage if: a) $u_i^e(g+ij) > u_i^e(g) \Rightarrow u_j^e(g+ij) < u_j^e(g)$ for all $ij \notin g$; b) $u_i^e(g) \geq u_i^e(g - ij)$ for all $ij \in g$.*

A pairwise stable network is here interpreted as an information structure that results from long-run information sharing arrangements, with the property that no additional arrangement occurs and no existing arrangement is discontinued.

INFORMATION SHARING WITH UNCORRELATED SIGNALS

We start by studying the case of uncorrelated signals, i.e. of $p_n = 0$. Formally, this requires that signal errors are negatively correlated, and that this correlation exactly outweighs the correlation induced by the state of the world: $u_n = -t$. Although this is a special case of signal's correlation, it is of interest here for two reasons. First, it allows us to better understand in the next section the role of signals' correlation, and of the associate strategic inference, on link formation. Second, uncorrelated signals are of interest in a model where the state of the world is the sum (or the average) of agents' signals, as in Gal-Or (1983), Hagenbach and Koessner (2010) and most papers dealing with common value problems in auctions, as, for instance, Levine (2004).

Uncorrelated signals imply that $k_1^{ig} = \frac{t}{p_s}$ and $k_2^{ig} = 0$. Equilibrium coefficients simply as follows:

$$\alpha = -\frac{\lambda_a + \gamma_{a\theta}\mu_\theta}{2\gamma_a + \gamma_{aA}(n-1)}; \tag{21}$$

$$\beta^g_{ih} = -\frac{\gamma_{a\theta}\frac{t}{p_s}}{2\gamma_a + \gamma_{aA}(n^g_h - 1)}, \ \forall i \in N^g_h. \tag{22}$$

The difference $[u^e_i(g') - u^e_i(g)]$ in (20) simplifies as follows:

$$\gamma_A p_s \left[\sum_{k\in N}(B^{g'}_{ik})^2 - \sum_{k\in N}(B^g_{ik})^2\right] + \gamma_{A\theta} t \left[\sum_{k\in N} B^{g'}_{ik} - \sum_{k\in N} B^g_{ik}\right] - \gamma_a p_s \left[\sum_{h\in N^g_i}(\beta^{g'}_{ih})^2 - \sum_{h\in N^g_i}(\beta^g_{ih})^2\right] \tag{23}$$

Since the notion of pairwise stability is defined link-wise, we are interested in the incentives to either form or sever a given link ij. This leads us to study the change in expected payoff when moving from a network g to a network $g' = g + ij$. Using (22), we note that for such networks we have $B^{g'}_{ik} = B^g_{ik}$ for all $k \neq i, j$, since $n^{g'}_k = n^g_k$. This implies that:

$$\sum_{k\in N}(B^{g'}_{ik})^2 - \sum_{k\in N}(B^g_{ik})^2 = (B^{g'}_{ii})^2 - (B^g_{ii})^2 + (B^{g'}_{ij})^2 - (B^g_{ij})^2 \tag{24}$$

and

$$\sum_{k\in N} B^{g'}_{ik} - \sum_{k\in N} B^g_{ik} = B^{g'}_{ii} - B^g_{ii} + B^{g'}_{ij} - B^g_{ij}. \tag{25}$$

Also, using the fact that $n^{g'}_i - 1 = n^g_i$ and $n^{g'}_j - 1 = n^g_j$ from (22) we obtain:

$$B^{g'}_{ii} = -\frac{n^g_i \gamma_{a\theta}\frac{t}{p_s}}{2\gamma_a + \gamma_{aA} n^g_i} \tag{26}$$

$$B^g_{ii} = -\frac{(n^g_i - 1)\gamma_{a\theta}\frac{t}{p_s}}{2\gamma_a + \gamma_{aA}(n^g_i - 1)} \tag{27}$$

$$B^{g'}_{ij} = -\frac{n^g_j \gamma_{a\theta}\frac{t}{p_s}}{2\gamma_a + \gamma_{aA} n^g_j} \tag{28}$$

$$B^g_{ij} = -\frac{(n^g_j)\gamma_{a\theta}\frac{t}{p_s}}{2\gamma_a + \gamma_{aA}(n^g_j - 1)} \tag{29}$$

Condition (23) can now be written as:

$$\begin{aligned}
&\gamma^2_{a\theta}\frac{t^2}{p_s}\gamma_A \left[\frac{(n^g_i)^2}{(2\gamma_a + \gamma_{aA} n^g_i)^2} - \frac{(n^g_i - 1)^2}{(2\gamma_a + \gamma_{aA}(n^g_i - 1))^2} + \frac{(n^g_j)^2}{(2\gamma_a + \gamma_{aA} n^g_j)^2} - \frac{(n^g_j)^2}{(2\gamma_a + \gamma_{aA}(n^g_j - 1))^2}\right] \\
&+ \gamma_{a\theta}\frac{t^2}{p_s}\gamma_{A\theta}\left[-\frac{n^g_i}{2\gamma_a + \gamma_{aA} n^g_i} + \frac{n^g_i - 1}{2\gamma_a + \gamma_{aA}(n^g_i - 1)} - \frac{n^g_j}{2\gamma_a + \gamma_{aA} n^g_j} + \frac{n^g_j}{2\gamma_a + \gamma_{aA}(n^g_j - 1)}\right] - \\
&\gamma^2_{a\theta}\frac{t^2}{p_s}\gamma_a\left[\frac{1}{(2\gamma_a + \gamma_{aA} n^g_i)^2} + \frac{1}{(2\gamma_a + \gamma_{aA} n^g_j)^2} - \frac{1}{(2\gamma_a + \gamma_{aA}(n^g_i - 1))^2}\right].
\end{aligned} \tag{30}$$

We will study pairwise stable networks in different classes of games, corresponding to different assumptions on the four key parameters in (30): γ_A, γ_a, γ_{aA} and $\gamma_{A\theta}$. These classes include the classical games outlined in our examples: Cournot competition, Beauty contest, Public good contribution game. We start with the case $\gamma_A = \gamma_{A\theta} = 0$, covering Cournot competition as a special case. From now on we will denote by $\mu \equiv \frac{\gamma_{aA}}{\gamma_a}$ the relative strategic interdependence in the game.

Proposition 3 *Let $p_n = 0$ and $\gamma_A = \gamma_{A\theta} = 0$. If $\mu < 0$ the unique pairwise stable network is the complete network. For $1 > \mu > \frac{2}{1+\sqrt{2}}$, then the set of pairwise stable networks is characterized as follows: for all $S \subseteq N$, all networks in which nodes in S are isolated, and all other nodes are organized in fully connected components of increasing sizes. For $0 < \mu < \frac{2}{1+\sqrt{2}}$ all pairwise stable networks contain at most one isolated node ($|S| \leq 1$ in the above characterization).*

Proof. We first note that if $\mu < 0$, that is if $\gamma_{aA} > 0$, direct inspection of (30) shows that under our assumption that $\gamma_a + (n-1)\gamma_{aA} < 0$, each agent has an incentive to link with all other agents independently of their degree. This implies that the complete network is the unique pairwise stable architecture. Let us then consider the case $\mu > 0$. We will show that each component in a pairwise stable network g must be fully connected, and then that components of equal size are incompatible with pairwise stability. We need to prove the following preparatory lemma.

Lemma 1 *Let $p_n = 0$. Let also $\gamma_A = \gamma_{A\theta} = 0$ and $\gamma_{aA} < 0$. Let g and g' be such that $g' = g + ij$. Then:*

1) if $n_j^g < n_i^g$ then $u_i^e(g') > u_i^e(g)$;
2) if $n_i^g = n_j^g \geq 2$ then $u_i^e(g') > u_i^e(g)$;
3) if $n_i^g = n_j^g = 1$ then $u_i^e(g') > u_i^e(g)$ iff $\mu < 2(\sqrt{2}-1)$;
4) there exists degree levels $f_\mu(n_i^g)$ and $F_\mu(n_i^g)$ such that: $n_j^g > f_\mu(n_i^g) \iff u_i^e(g') < u_i^e(g)$ and $n_j^g < F_\mu(n_i^g) \iff u_i^e(g') > u_i^e(g)$. Moreover, $f(m) < F(m)$ for all $m \geq 2$.

Proof of Lemma 1. Using (26-29) we see that $u_i^e(g') - u_i^e(g) > 0$ iff

$$\frac{1}{(2+\mu(n_i^g - 1))^2} < \frac{1}{(2+\mu(n_i^g))^2} + \frac{1}{(2+\mu(n_j^g))^2}. \tag{31}$$

Since $n_j^g < n_i^g$ implies $n_j^g \leq n_i - 1$, the above condition is always satisfied for $n_j^g < n_i^g$. Let us then study the case $n_i^g = n_j^g = m$. Using again (26-29)we obtain

$$B_{ii}^{g'} = B_{ij}^{g'} = \frac{m\gamma_{a\theta}\frac{t}{p_s}}{2\gamma_a + \gamma_{aA}m} \tag{32}$$

$$B_{ii}^{g} = B_{ij}^{g} = \frac{(m-1)\gamma_{a\theta}\frac{t}{p_s}}{2\gamma_a + \gamma_{aA}(m-1)} \tag{33}$$

and we can write the difference $u_i^e(g') - u_i^e(g)$ as follows:

$$\gamma_{a\theta}^2 \frac{t^2}{p_s^2}\gamma_A p_s \left[\frac{2m^2}{(2\gamma_a+\gamma_{aA}m)^2} - \frac{2m^2-2m+1}{(2\gamma_a+\gamma_{aA}(m-1))^2}\right] + \gamma_{a\theta}^2 \frac{t^2}{p_s^2}\gamma_{A\theta} t \left[\frac{2m}{(2\gamma_a+\gamma_{aA}m)} - \frac{2m-1}{(2\gamma_a+\gamma_{aA}(m-1))}\right] - \gamma_{a\theta}^2 \frac{t^2}{p_s^2}\gamma_a p_s \left[\frac{2}{(2\gamma_a+\gamma_{aA}m)^2} - \frac{1}{(2\gamma_a+\gamma_{aA}(m-1))^2}\right]. \quad (34)$$

Assuming $\gamma_A = \gamma_{A\theta} = 0$, the above expression is strictly positive iff:

$$2(2\gamma_a+\gamma_{aA}(m-1))^2 > (2\gamma_a+\gamma_{aA}m)^2. \quad (35)$$

Assuming $\gamma_{aA} < 0$, dividing both sides by γ_a^2 and letting $\mu = \frac{\gamma_{aA}}{\gamma_a}$, we obtain:

$$2(2+\mu(n-1))^2 - (2+\mu n)^2 > 0.$$

The two roots of the LHS of the above inequality are:

$$\frac{4\mu(\mu-1) \pm \sqrt{8\mu^4}}{2\mu^2},$$

yielding the following larger root:

$$\frac{4(\mu-1)+2\mu^2\sqrt{2}}{2\mu^2}.$$

The larger root is smaller than n for:

$$(m-\sqrt{2})\mu^2 - 2\mu + 2 > 0,$$

which is always satisfied for $m \geq 2$.

When $m = 1$, from condition (35) we obtain that the difference $u_i^e(g') - u_i^e(g)$ is positive if and only if $\mu < 2(\sqrt{2}-1) < 1$.

Finally, we look at the incentives for i to link with j when $n_j^g > n_i^g$. Let

$$g_\mu(m) \equiv \frac{1}{(2+\mu m)^2}.$$

We define the function $f_\mu(m)$ as follows:

$$f_\mu(m) = g_\mu^{-1}(g_\mu(m-1) - g_\mu(m)).$$

The value $f_\mu(m)$ is the maximal degree that a node k can have for a node of degree m to wish to form a link with k. Similarly, we define a function $F_\mu(m)$ that identifies the maximal degree that a node k can have in order for a node of degree m to maintain a link with k:

$$F_\mu(m) = g_\mu^{-1}(g_\mu(m-2) - g_\mu(m-1)) + 1.$$

Algebraic computations show that $f_\mu(m) > F_\mu(m)$ for all $m > \frac{2(\mu-1)}{\mu}$. Since the condition $m > \frac{2(\mu-1)}{\mu}$ is always satisfied when $m \geq 2$, this concludes the proof of the Lemma. ■

We are now ready to prove proposition 3. We proceed by contradiction, assuming that two agents i and j who are not connected and belong to the same component. By the previous lemma, stability requires that $n_i^g > n_j^g$. This implies that there exist an agent k such that $ik \in g$ and $ij \notin g$. The proof goes by showing that stability of g requires that $n_k^g > n_i^g$. By the previous lemma, pairwise stability requires that $n_i^g \leq F(n_j^g)$, or equivalently that $n_i^g - 1 \leq F_\mu(n_j^g) - 1$. Since $n_k^g \leq n_i^g - 1$, we obtain $n_k^g \leq F_\mu(n_j^g) - 1 < f_\mu(n_j^g)$, where we have used the previous lemma for the last inequality. In words, this means that j has an incentive to link to k. Let us now show that also k wants to link to j. Pairwise stability applied to the link ik implies that $n_i^g \leq F_\mu(n_k^g)$. Also, from the previous lemma we have $F_\mu(n_k^g) - 1 < f_\mu(n_k^g)$. These, together with the fact that $n_j^g \leq n_i^g - 1$ imply that $n_j^g < f(n_k^g)$, which means that k has a strict incentive to link to j, contradicting stability of g. If $n_k^g = n_i^g$, then agent j wants to form a link with k, since $n_k^g = n_i^g \leq F_\mu(n_j^g) < f_\mu(n_j^g)$, where the last inequality comes from lemma 1. We have therefore proved that $d_k > n_i^g$. Applying the same steps to the new pair i and k, we conclude that there must exist some other agent l such that $n_l^g > n_k^g$. Since the network is finite, this recursive argument implies a contradiction. The fact that components must have increasing sizes, and that these sizes are determined by the function f_μ, comes directly from lemma 1, while the fact that only one singleton can appear in a stable network when $\mu < \frac{2}{1+\sqrt{2}}$ comes from the fact that in this range of values two isolated agents have mutual incentives to form a link. ■

It is worth commenting on the way in which incentives to link vary with the degrees of the involved nodes, as described by the two functions f_μ and F_μ in Lemma 1. In particular, we have found that both f_μ and F_μ are increasing, meaning that the more connected a node i is, the larger the maximal degree of a node j that i would accept to link to. It is also immediate to check that the incentives of i to link with j are decreasing in the degree of j. Thus, the requirement of increasing sizes of the fully components of stable networks has the scope of discouraging members of the smaller component to link to members of the larger components.

The next corollary applies proposition 3 to the model of Cournot Oligopoly with i.i.d. signals (as the one studied in Gal-Or (1985)).

Corollary 1 *In a Cournot Oligopoly with i.i.d. signals about the demand intercept, information sharing is organized in groups of firms, and within each group all information is universally disclosed. Groups have increasing sizes, and size differences increase with the degree of products differentiation.*

While the results in Gal-Or (1985) indicate that in the unique equilibrium firms do not exchange private information we find that full or at least some level of information sharing is consistent with equilibrium. This difference mainly arises from the different technology of information sharing that we adopt in our model and that allows for bilateral agreements. Moreover whenever information is shared in equilibrium this happens in very specific form, with groups of firms sharing all their information within the group and not share information with firms outside of the group. In the next proposition we turn to the case in which payoffs depend directly on the variance of opponents' actions in equilibrium. It shows that when such variance is beneficial, then the complete network is still the unique pairwise stable network under strategic complementarity.

Proposition 4 *Let $\gamma_A > 0$, $\gamma_{A\theta} = 0$ and $-\frac{\gamma_a}{n-1} > \gamma_{aA} > 0$. Then the complete network is the unique pairwise stable network.*

Proof. From direct inspection of (30) we see that both terms in squared brackets are positive, which, together with the assumptions that $\gamma_a < 0$ and $\gamma_A > 0$ imply that all links form. ■

The above result is explained by the fact that when complementarities are small in the sense of the range assumed for γ_a, then an additional link formed by player i always increases the volatility of her own equilibrium strategy and of the aggregate of all other players' equilibrium strategies. When this increased volatility has a positive effect on i's payoff ($\gamma_A > 0$), then these two positive effect sums up with the positive effect of i's own increased variability (the last term of (30)).

Let us now turn to the case $\gamma_A < 0$, in which opponents' volatility is detrimental to an agent's payoffs . Here, an additional link ij resulting in increased variance in opponents' strategies may not always increase i's payoff even if strategies are complements. One class of games in which strategies are complements and $\gamma_A < 0$ is Keynes' Beauty Context, where if $v < 1$ we have $\gamma_{aA} = \dfrac{2(1-v)}{n-1} > 0$ and $\gamma_A = -\dfrac{1-v}{(n-1)^2} < 0$. The next proposition shows that in this game, the negative effect of opponents' increased volatility never outweighes one's own, and all links form in equilibrium.

Proposition 5 *Consider the Beauty Context with $0 < v < 1$. The complete network is the unique pairwise stable information structure.*

Proof. A node of degree x has an incentive to link to a node of degree y iff the following expression is positive (see (30)):

$$\gamma_A \left[\frac{x^2}{(2\gamma_a + \gamma_{aA}x)^2} - \frac{(x-1)^2}{(2\gamma_a + \gamma_{aA}(x-1))^2} + \frac{y^2}{(2\gamma_a + \gamma_{aA}y)^2} - \frac{y^2}{(2\gamma_a + \gamma_{aA}(y-1))^2} \right]$$
$$-$$
$$\gamma_a \left[\frac{1}{(2\gamma_a + \gamma_{aA}x)^2} + \frac{1}{(2\gamma_a + \gamma_{aA}y)^2} - \frac{1}{(2\gamma_a + \gamma_{aA}(x-1))^2} \right] > 0. \quad (36)$$

with the following expression for the various parameters of the payoff function: $\gamma_a = -1$; $\gamma_A = -\frac{(1-v)}{(n-1)^2}$; $\gamma_{aA} = \frac{2(1-v)}{n-1}$. Note first that for $x = 1$ and $y = 1$ incentives are positive iff:

$$\frac{(n^2 - 2n + v)(n^2 - 2 + 4v - 2nv - v^2)}{(4(n-1)^2(n + v - 2)^2)} > 0. \tag{37}$$

It can be checked that for $0 < v < 1$ (37) holds for all values of $0 < x < n$ and $0 < y < n$. We then compute the derivative of expression (36) with respect to x and to y. Both derivatives are positive. This implies that a node of degree x has an incentive to link to a node of link y for all $x, y < n$, which implies the result. Exact computations of the derivatives involve long expressions and are available upon request. ■

Although the restrictions imposed by the Beauty Contest on the parameters ($2\gamma_A = -\frac{\gamma_{aA}}{n-1}$) ensure that the incentives to form a new link always remain positive, once such restrictions are dropped (and, in particular, γ_A can grow in magnitude fixing the other parameters) incomplete networks may arise even in the presence of strategic complementarities. In the next example we study a variation of Keynes' Beauty contest , in which agents try to match the state of the world and to exceed the average of their opponents' strategies by a factor k. For this game we show that incomplete networks arise in equilibrium.

Example 2 *Consider an economy with 10 agents, each having the following payoff function:*

$$u_i(a_i, A_i, \theta) = -v(a_i - \theta)^2 - (1 - v)(a_i - k\frac{A_i}{n-1})^2.$$

Set $\mu = -\dfrac{2}{19}$ *and* $\gamma_a = -1$*, so that our condition* $\gamma_a + (n-1)\gamma_{aA} < 0$ *holds. These assumptions imply* $v = 0.53$ *and* $\gamma_A = -0.0058$*. Computations based on condition (36) show that for* $k = 1$*, any two any nodes with arbitrary degree* $x < 10$ *always have an incentive to form a link. This is simply a consequence of proposition 5 where the complete network is shown to be the unique pairwise stable network in the Beauty Contest.*

Consider now values of $k > 1$*, setting all other parameters as above, we obtain* $v = \frac{19k-9}{19k}$ *and then* $\gamma_A = -0.0058 \cdot k^2$*. Computations show that for k large enough, incomplete networks can be pairwise stable. For example, when* $k = 5$ *we obtain that 1) a regular network with average degree of 8 is pairwise stable; 2) networks with one fully connected component of 9 nodes and one isolated node or two fully connected components of two and eight nodes are pairwise stable.*

We finally turn the case in which both parameters γ_A and $\gamma_{A\theta}$ are possibly non null. Here we will only focus on games with orthogonal strategies ($\gamma_{aA} = 0$), and and concentrate on the interplay of the parameters γ_A, $\gamma_{a\theta}$ and $\gamma_{A\theta}$. As a specific case we have the public good game with linear benefits and quadratic costs, where $\gamma_{A\theta}$ is equal to 1 and $\gamma_A = 0$.

Proposition 6 *Let* $\gamma_{aA} = 0$ *and* $\gamma_{a\theta} > 0$*.*

1. *When* $\gamma_A = 0$*, the unique pairwise stable network is either the complete network (iff* $2\gamma_{A\theta} + \gamma_{a\theta} > 0$*) or the empty network (iff* $2\gamma_{A\theta} + \gamma_{a\theta} < 0$*).*

2. *When* $\gamma_A > 0$*, then:*

   (a) *if* $2\gamma_{A\theta} + \gamma_a\theta > 0$ *then the complete network is the unique pairwise stable architecture;*

   (b) *if* $2\gamma_{A\theta} + \gamma_a\theta < 0$ *then there exists* $m \leq n$ *such that the set of all pairwise stable networks consists of all networks made of a fully connected component of size* q *and* $n - q$ *isolate nodes, with* $q \geq$*, together with the empty network.*

3. *When* $\gamma_A < 0$*, then:*

   (a) *if* $2\gamma_{A\theta} + \gamma_a\theta > 0$*, either* $1 < \frac{2\gamma_{A\theta}+\gamma_{a\theta}}{\gamma_{a\theta}}$*, in which case the empty network is the unique pairwise stable architecture, or otherwise no pairwise stable network exists;*

   (b) *if* $2\gamma_{A\theta} + \gamma_a\theta < 0$*, only the empty network is pairwise stable.*

Proof. From condition (30) we obtain that agent i finds it profitable to form link $ij \notin g$ if and only if the following condition holds:

$$\frac{\gamma_{a\theta}\gamma_A}{4\gamma_a^2}(2n_i^g - 1) - \frac{\gamma_{aA}}{2\gamma_a} - \frac{\gamma_{A\theta}}{4\gamma_a} > 0$$

where we have used the assumption that $\gamma_{a\theta} > 0$. This yells the following:

$$\frac{\gamma_A}{\gamma_a}(2n_i^g - 1) < \frac{2\gamma_{A\theta} + \gamma_{a\theta}}{\gamma_{a\theta}}.$$

It is immediate that when $\gamma_A > 0$ and $\frac{2\gamma_{A\theta}+\gamma_{a\theta}}{\gamma_{a\theta}} > 0$ the above is always satisfied. When $\gamma_A > 0$ and $\frac{2\gamma_{A\theta}+\gamma_{a\theta}}{\gamma_{a\theta}} < 0$ web have that the above is satisfied for $n_i^g > \frac{2\gamma_{A\theta}+\gamma_{a\theta}}{\gamma_{a\theta}}\frac{\gamma_a}{\gamma_A} \equiv m > 0$. In this case the empty network is trivially stable, as is the complete network. The only other stable architecture must have all nodes with positive degree with degree larger than m, and all such nodes must be linked to each other.

When $\gamma_A < 0$, we instead have that the above condition is never satisfied when $\frac{2\gamma_{A\theta}+\gamma_{a\theta}}{\gamma_{a\theta}} < 0$ (from which the empty network as unique pairwise stable one), while if $\frac{2\gamma_{A\theta}+\gamma_{a\theta}}{\gamma_{a\theta}} > 0$ the relevant condition for agent i to form a link is the following:

$$(2n_i^g - 1) < \frac{2\gamma_{A\theta} + \gamma_{a\theta}}{\gamma_{a\theta}}\frac{\gamma_a}{\gamma_A}.$$

Here, only agents with a low enough degree would form a link. If the threshold degree m is less than zero, then the empty network is the unique pairwise stable network; if not, no pairwise stable network exists. To see this, note that two nodes who are linked in a stable network must have degree less than m. But in this case they wish to form a link to the agents to which they are not linked. If these agents have degree less than m they also want to link, then we contradict the stability of the network. If they do not wish to link, then they must have a degree which is larger than m, in which case they wish to sever a link. Finally, the empty network is not stable since two nodes of degree zero wish to form a link. ■

The parameter γ_A measures the effect on payoffs of a change in the variability of opponents' strategies. In the class of games covered by proposition 6, this change is always positive as a result of one additional link, and increasing in the degree of the agent forming the new link. Note also that when $\gamma_{A\theta} > 0$, then the effect on expected utility of a new link is always positive. This means that if an agent benefits from the covariance of opponents' strategies and the state of the world ($\gamma_{A\theta} > 0$, then the complete network always forms. In contrast, if such covariance is detrimental to utility, then links form only if the (positive) effect of the increased opponents' volatility is large enough, that is for large enough degrees. As an illustration of this second case, the next example studies a game in which agents wish to guess the state of the world, with a reward that increases the worse is the opponents' guess.

Example 3 *Each agent i has the following utility from her guess a_i and the opponents' guess A_i:*

$$u_i = -(a_i - \theta) + v(\frac{A_i}{n-1} - \theta).$$

We have $\gamma_a = -1$, $\gamma_{A\theta} = -2\frac{v}{(n-1)}$, $\gamma_{a\theta} = 2$, and $\gamma_A = \frac{v}{(n-1)^2}$. From condition (30) and the fact that $\gamma_{aA} = 0$, agent i forms a link ij if and only if the following condition holds:

$$-\frac{v}{(n-1)^2}(2n_i^g - 1) < 1 - 2\frac{v}{(n-1)}$$

or

$$(2n_i^g - 1) > 2(n-1) - \frac{(n-1)^2}{v}$$

For $v = 2$ and $n = 3$ we have that agent i wishes to form the link ij if and only if $n_i^g > 1.5$. This implies that there are three pairwise stable architectures in this example: the empty network, the complete network, and a component of two connected nodes and one isolated node. For $v = 4$ the complete network remains the unique stable network, together with the empty network. If v increases further, the empty network remains the unique stable architecture.

Proposition 6 has the following corollary for the case of the public good game, in which $\gamma_A = 0$ and $2\gamma_{A\theta} + \gamma_{a\theta} = 3$.

Corollary 2 *In the public good game with linear benefits and quadratic costs, the unique pairwise stable network is the complete network.*

INFORMATION SHARING WITH CORRELATED SIGNALS

We now turn to the case of correlated signals, that is $p_n > 0$. Here equilibrium computations become quite complex due to the potential asymmetry of network structures and of the associated Bayesian Nash equilibria. For this reason we will not provide a full characterization of pairwise stable networks, but rather investigate the effect of signals' correlation, and in particular of *conditional* correlation, on the incentives to share information. In particular we will show that enough correlation guarantees that some positive amount of information is always shared in equilibrium - in the present terminology, that the empty network fails to be pairwise stable. As we shall see, this is due to the strategic advantage that the bilateral information sharing provides in the form of a better inference of other firms' actions.

We start with the case of strategic substitutes ($\mu > 0$), assuming that only the volatility of one's own strategy is payoff relevant. This includes Cournot competition as a special case.

Proposition 7 *Let $\gamma_A = \gamma_{A\theta} = 0$ and $\mu > 0$. Then:*

1) The complete network is always pairwise stable;

2) The empty network is not pairwise stable under the following conditions: i) $\mu < \frac{2}{3}$; ii) $\frac{2}{3} < \mu < \frac{2}{1+\sqrt{2}}$ and either $p_n < p_n^{**}$ or $p_n > p_n^{**}$ and $n > n^{**}$, where both p_n^{**} and n^{**} are finite and positive; iii) $\mu > \frac{2}{1+\sqrt{2}}$, $p_n > p_n^*$ and $n > n^*(p_n)$, where both p_n^* and $n^*(p_n)$ are finite and positive.

Proposition 7 is to be interpreted as a result about the occurrence of information sharing in equilibrium. First, the universal sharing of all available information is stable against revisions by any pair of players, even when strategies are substitutes and independently of signals' correlation. Second, *all stable networks* involve some amount of information sharing, provided both signals' correlation and the number of players are not too small. This is in striking contrast with traditional conclusions about information sharing in Cournot oligopolies, where market competition is found incompatible with sharing unless goods are strongly differentiated (that is, unless strategies are weakly substitutes or complements - see seminal works by Novshek and Sonnenschein, 1982, Vives, 1985, Kirby, 1988, Li, 1995). The intuition behind the result of proposition 7 is as follows: even when the game is one of strategic substitutes, still two agents may have incentives to share their own information if this provides them with a substantial refinement about opponents' behaviour. The key to the result is that this refinement (due to signals' conditional correlation) comes with a limited increase in the equilibrium covariance of strategies, since it does not imply the transmission of one's own signal to more than one other agent (the other end of the new link). The effect of such refinement on the incentive to share increases with the number of agents, from which the requirement on n in the proposition. Proposition 7 is rephrased in the I.O. terminology in the following corollary.

Corollary 3 *In the Cournot Oligopoly with demand uncertainty, some amount of information sharing, in the form of bilateral agreements between firms, is consistent with non cooperative market competition, even when products are perfect substitutes and costs are linear, and is a feature of all equilibria when a large (but finite) number of firms receive strongly correlated signals.*

Understanding the effect of correlation in general is, however, hard. More correlation will in fact improve both the precision on unobserved signals before and after the additional link is formed. Intuitively, when an agent has little information, the gains (in terms of refined expectations) from one additional piece is substantial, leading to point 2) of proposition 7. However, as the stock of one's information builds up, the incentives to access more information may decrease. There may be therefore cases in which some information, but not all, is shared in equilibrium. The next four-player example fully characterize the set of pairwise stable networks, and provides some clearer insight on how correlation affects the incentives to share information in different networks.

Example 4 *Let $n = 4$, $\mu = 1$ and $p_s = 1$. The pairwise stable networks in the various ranges of the correlation parameter p_n are:*

- $p_n < 0.62$*: the complete network, the empty network and one complete component of 3 nodes;*
- $0.62 < p_n < 0.71$*: the complete and the empty networks.*
- $0.71 < p_n < 0.75$*: the complete network, the empty network and one complete component of 3 nodes;*
- $p_n > 0.75$*: the complete network and one complete component of 3 nodes;*

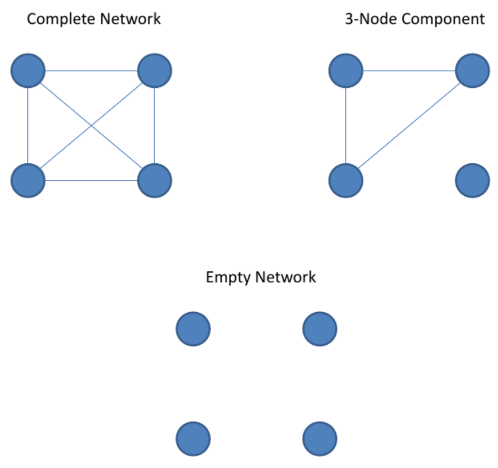

Figure 1:
Pairwise Stable Networks with 4 nodes.

For low levels of signals' correlation ($p_n < 0.62$), the architecture of pairwise stable networks is in accordance with our proposition 3, dealing with the case of no correlation: fully connected components and, possibly, isolated nodes. Here the following incentives are at work: two players do not form a link when isolated, and form a link when they share the same degree. These incentives result in the empty and the complete network being stable. Moreover, players in the 3-node component have an incentive to form a link with the remaining isolated player who, in contrast, does not have such incentive. This is consistent with our discussion of proposition 3, where we argued that the incentive for i to link with j increase with I's degree and decrease with j's degree. As correlation increases ($0.62 < p_n < 0.71$), so does the incentive of the isolated player to gain information about the opponents' behaviour by forming one additional link, and thus leads to the instability of the network containing the 3-node component. However, further increases of correlation ($0.71 < p_n < 0.75$) decrease the incentive of each members of the 3-node component to link with the isolated node, whose behavior is now predicted with high precision thanks to the high conditional correlation of signals. The isolated players remains thereby excluded from sharing. For large enough levels of correlation ($p_n > 0.75$), even single isolated players would form a link, and all pairwise stable networks display some amount of information sharing, consistently with point 2) in proposition 7. Summing up, signals' correlation creates incentives to share information, but only up to the point at which well connected (and informed) agents find it profitable not to further reveal their own private information in exchange for unobserved signals on which their already have a very precise expectation.

We then turn to the Beauty Contest, where strategies are complements and payoffs also depend on the volatility of opponents' equilibrium strategies. We prove a partial counterpart of proposition 3, showing that the universal sharing of information is always pairwise stable, and that all stable networks involve some amount of information sharing.

Proposition 8 *In the Beauty context game with strategic complements ($0 < v < 1$):*

1. *the complete network is pairwise stable;*
2. *the empty network is not pairwise stable.*

Finally, we study the case in which also the covariance of equilibrium strategies with the state of the world affects payoffs. We focus here on the public good game with linear benefits, for which we provide a full characterization of pairwise stable networks. The intuition is similar to the one behind corollary 2: both one's own equilibrium variance and opponents' covariance with the state of the world are beneficial, and both increase as a result of one additional link, resulting in the universal sharing of all information.

Proposition 9 *In the public good game with linear benefits and quadratic costs the complete network is the unique pairwise stable architecture.*

CONCLUSIONS

We have studied the incentives to bilaterally share information of agents playing a linear quadratic game, and which stable networks result from these incentives. Our main contribution compared to previous literatures has been to frame the sharing problem in the general linear quadratic model, and to allow agents to bilaterally share information. In our contest with identical agents, we have focused on the ex-ante commitment to truthfully reveal information, and we have studied under which conditions agents make such commitments. As we have shown, the general linear quadratic formulation is rich enough to generate non trivial network structures even in games with strategic complements, where incentives to share are strong. Our analysis has particularly focused on the role of signals' correlation in shaping incentives to share, and how and when incomplete network structure may arise in equilibrium. Our analysis of sharing has demanded a characterization of the equilibrium use of information in networks, which extends previous work on Bayesian equilibrium in linear quadratic games to the case of networked information structures.

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APPENDIX

Proof of Proposition 7

Proof of point 1: The proof is organized in several steps, and goes by studying the difference in expected profits of two firms, 1 and 2, in the complete networks g^c and in the network $g^{-12} \equiv \{g^c - 12\}$. We first compute equilibrium strategies in g^c. The updating coefficient in g^c is for every i:

$$k_1^{ig^c} = \frac{t}{p_s + (n-1)\,p_n}. \tag{38}$$

We obtain the following common equiibrium coefficient:

$$\beta^{g^c} = -\frac{t\gamma_{a\theta}}{(p_s + (n-1)\,p_n)\,(2\gamma_a + \gamma_{aA}\,(n-1))}.$$

For $g^{-12} \equiv \{g^c - 12\}$, the updating coefficients are:

$$\begin{aligned} k_1^{ig^{-12}} &= \frac{t}{p_s + (n-2)\,p_n}, \quad k_2^i = \frac{p_n}{p_s + (n-2)\,p_n}, \quad i = 1, 2 \\ k_1^{ig^{-12}} &= \frac{t}{p_s + (n-1)\,p_n}, \quad \forall i \geq 3 \end{aligned} \tag{39}$$

We obtain the following equilibrium coefficients for firms 1 and 2:

$$\begin{aligned} \beta_{11}^{g^{-12}} &= \beta_{22}^{g^{-12}} = -\frac{t\gamma_{a\theta}}{2\gamma_a\,(p_s + (n-2)\,p_n) + \gamma_{aA}\,((n-2)\,p_s + (5 + n\,(n-4))\,p_n)} \\ \beta_{1j}^{g^{-12}} &= \beta_{2j}^{g^{-12}} = \\ &= -\frac{t\,(2\gamma_a + (n-2)\,\gamma_{aA})\,\gamma_{a\theta}}{(2\gamma_a + (n-1)\,\gamma_{aA})\,(2\,(p_s + (n-2)\,p_n)\,\gamma_a + \gamma_{aA}\,((n-2)\,p_s + (5 + n\,(n-4))\,p_n))}, \ \forall j \geq 3 \end{aligned}$$

From (20), we can express the difference $u^e(g^c) - u^e(g^{-12})$ in the expected profits of firm 1 (and, by symmetry, of firm 2) in g^c and in g^{-12} as proportional to:

$$\begin{aligned} n \cdot \left(\beta_{ij}^{g^c}\right)^2 (p_s + (n-1)\,p_n) - \left(\left(\beta_{11}^{g^{-12}}\right)^2 + (n-2)\left(\beta_{1j}^{g^{-12}}\right)^2\right) p_s - \\ - (n-2)\,\beta_{1j}^{g^{-12}}\left(2\beta_{11}^{g^{-12}} + (n-3)\,\beta_{1j}^{g^{-12}}\right) p_n. \end{aligned} \tag{40}$$

Plugging in the values of the β coefficients, we obtain the following expression:

$$\frac{(p_s - p_n)\,t^2\left(4\,(p_s + (n-2)\,p_n)\,(1 + (n-3)\,\mu) + ((7 + (n-6)\,n)\,p_s + (n\,(19 + (n-8)\,n) - 16)\,p_n)\,\mu^2\right)\gamma_{a\theta}^2}{(p_s + (n-1)\,p_n)\,(2 + (n-1)\,\mu)^2\,(2p_s + 2\,(n-2)\,p_n + (n-2)\,p_s\mu + (5 + (n-4)\,n)\,p_n\mu)^2\,\gamma_a^2} \tag{41}$$

The denominator of the above equation is always strictly positive for all admissible values of the parameters; moreover the sign is the same as the sign of the following expression:

$$4\left(p_s+(n-2)\,p_n\right)\left(1+(n-3)\,\mu\right)+ \\ +\left((7+(n-6)\,n)\,p_s+\left(n\,(19+(n-8)\,n)-16\right)p_n\right)\mu^2 \tag{42}$$

We divide it in two terms. The first, $4\left(p_s+(n-2)\,p_n\right)\left(1+(n-3)\,\mu\right)$, is always positive: indeed by assumption $\mu>0$ and the proof follows directly; it can be directly verified that the second term is positive for $n\geq 5$. Therefore (42) could be negative only for $n=3$ and $n=4$. But for $n=3$ (42) becomes $4\left(p_s+p_n\right)-2\left(p_s+2p_n\right)\mu^2$ and for $n=4$ (42) becomes $4\left(p_s+2p_n\right)(1+\mu)-\left(p_s+4p_n\right)\mu^2$ and, by the assumption that $0<\mu<1$, both terms are strictly positive.

Proof of point 2

We study the difference in expected profits of two agents, 1 and 2, in the empty network $g^{\varnothing}$ and in the network $g^{12}\equiv\{12\}$. The updating coefficients for $g^{\varnothing}$ are:

$$k_1^{ig^{\varnothing}}=\frac{t}{p_s},\quad k_2^{ig^{\varnothing}}=\frac{p_n}{p_s},\quad\forall i \tag{43}$$

from which we obtain the common coefficient of agents' equilibrium strategies:

$$\beta_{ii}^{g^{\varnothing}}=-\frac{\gamma_{a\theta}t}{2\gamma_a p_s+\gamma_{aA}\,(n-1)\,p_n},\quad\forall i$$

The updating coefficients for $g^{12}\equiv\{12\}$ are:

$k_1^{ig^{12}}=\frac{t}{p_s+p_n}$ for $i=1,2$

$k_1^{ig^{12}}=\frac{t}{p_s}$ for all $i\geq 3$

$k_2^{ig^{12}}=\frac{p_n}{p_s+p_n}$ for $i=1,2$

$k_2^{ig^{12}}=\frac{p_n}{p_s}$ for $i\geq 3$

We obtain the following equilibrium coefficients for agents 1 and 2:

$$\beta_{11}^{g^{12}}=\beta_{12}^{g^{12}}=\beta_{21}^{g^{12}}=\beta_{22}^{g^{12}}= \\ =-\frac{t\left(2p_s\gamma_a-\gamma_{aA}p_n\right)\gamma_{a\theta}}{4p_s\left(p_s+p_n\right)\gamma_a^2+2\left(p_s+p_n\right)\left(p_s+(n-3)p_n\right)\gamma_a\gamma_{aA}+p_n((n-3)\,p_s-(3n-5)\,p_n)\gamma_{aA}^2}.$$

From (20), we can express the difference in profits of agent 1 (and, by symmetry, of agent 2) in $g^{\varnothing}$ and in g^{12} as:

$$\left(\beta_{ii}^{g^{\varnothing}}\right)^2 p_s-2\left(\beta_{11}^{g^{12}}\right)^2\left(p_s+p_n\right). \tag{44}$$

Plugging in (44) the values of the β coefficients, recalling the definition of μ and letting $p\equiv p_s+p_n$ we obtain the following expression:

$$\frac{t^2\gamma_{a\theta}^2}{\gamma_a^2}\left[\frac{p_s}{\left(2p_s+(n-1)\,p_n\mu\right)^2}-\frac{2p\left(p_n\mu-2p_s\right)^2}{\left(4p_sp+2p\left(p_s+(n-3)\,p_n\right)\mu-p_n\left((3n-5)\,p_n-(n-3)\,p_s\right)\mu^2\right)^2}\right]$$

It can be shown that the denominator of the above expression is strictly positive. Its sign of is therefore the sign of the numerator of the above expression, which can be written in the following form:

$$a \cdot n^2 + b \cdot n + c \tag{45}$$

where

$$\begin{aligned} a &= (p_s - p_n)\, p_n^2 \mu^2 \left(4p_s p\,(\mu - 1) + \left(p_s^2 - 5p_s p_n + 2p_n^2\right)\mu^2\right) \\ b &= 2\,(p_s - p_n)\, p_n \mu \cdot \\ &\cdot \left(-8p_s^2 p + 4p_s p\,(2p_s + 3p_n)\,\mu + 2p_s\,(p_s - 8p_n)\, p\mu^2 - p_n\left(3p_s^2 - 11p_s p_s + 2p_n^2\right)\mu^3\right) \\ c &= (p_s - p_n)\left[2p_n^4\mu^4 + p_s p_n^3 \mu^2\left((44 - 21\mu)\,\mu - 36\right) + 4p_s^4\left(\mu\,(4 + \mu) - 4\right)\right. \\ &\left. -4p_s^3 p_n\,(\mu - 1)\left(3\mu\,(4 + \mu) - 4\right) + p_s^2 p_n^2 \mu\left(48 + \mu\left(\mu\,(32 + 9\mu) - 76\right)\right)\right] \end{aligned}$$

The proof continues now by studying the sign of (45).

We first note that the roots (n_-, n_+) of (45) are real (since $b^2 - 4ac \geq 0$), distinct and finite as long as $a \neq 0$. With this in mind, we now look for conditions under which expression (45) is concave. Such conditions will tell us whether the sign of (45) becomes negative for n large enough.

Lemma 2 *If $\mu < \frac{2}{1+\sqrt{2}}$ then (45) is concave. If $\mu > \frac{2}{1+\sqrt{2}}$ then there exists p_n^* such that for all $p_n > p_n^*$ (45) is concave, otherwise it is convex.*

Proof of Lemma 2: Note that concavity of (45) depends on the sign of term a in (45). This term is negative for $\mu < 0$. Moreover, the sign of a is the sign of the following term:

$$\left(4p_s p\,(\mu - 1) + \left(p_s^2 - 5p_s p_n + 2p_n^2\right)\mu^2\right). \tag{46}$$

Let us evaluate the roots of (46) as a function of p_n. We find:

$$\frac{4p_s\,(1 - \mu) + 5p_s\mu^2 \pm p_s\,(\mu - 2)\sqrt{4\,(1 - \mu) + 17\mu^2}}{4\mu^2} \tag{47}$$

Since the largest root yields a value which exceeds p_s, we only consider the smaller root denoted by p_n^*. Note here that the second derivative of (46) with respect to p_n is positive (so that a is a convex function of p_n). This directly implies that a is negative for all $p_n > p_n^*$. We then turn to the analysis of the root p_n^* in relation to the parameter μ. We show that if $\mu < \frac{2}{1+\sqrt{2}}$ then $p_n^* < 0$, implying that $a < 0$ for all parameters' values; moreover, when $\mu > \frac{2}{1+\sqrt{2}}$, we show that $p_n^* > 0$ and that p_n^* is increasing in μ. In this latter case, $a < 0$ for all values $p_n^* < p_n < p_s$.

Consider again the smaller root in (47):

$$p_n^* = \frac{4p_s\,(1 - \mu) + 5p_s\mu^2 + p_s\,(\mu - 2)\sqrt{4\,(1 - \mu) + 17\mu^2}}{4\mu^2}. \tag{48}$$

Expression (48) is null for the following values of μ:

$$\mu_- = 2\left(-1 - \sqrt{2}\right); \quad \mu_+ = \frac{2}{1 + \sqrt{2}}. \tag{49}$$

Moreover, the expression (48) is strictly increasing in μ for all values of μ in the range $(0,1]$. This implies that $p_n^* < 0$ for all $0 < \mu < \mu_+$, and that $p_s > p_n^* > 0$ for all $\mu_+ < \mu \leq 1$. ■

Having established conditions under which (45) is concave in n, we study its sign by establishing a few facts about the behaviour of (45) at the point $n = 2$.

Lemma 3 *At $n = 2$: i) expression (45) is negative for $\mu < \frac{2}{3}$, is positive for $\mu > \frac{2}{1+\sqrt{2}}$ and in the intermediate range is positive if and only if $p_n > \dfrac{p_s\left(4 - 4\mu - \mu^2\right)}{2\mu^2} \equiv p_n^{**}$. ii) Moreover, there exists $\hat{p}_n > 0$ such that (45) is increasing in n if $p_n < \hat{p}_n$ and $\mu > \frac{2}{1+\sqrt{2}}$, otherwise (45) is decreasing in n.*

Proof of Lemma 3. Point *i)* follows from direct computation, and is consistent with Proposition 4.4 in Raith (1996) for the specific case of Cournot oligopoly, setting $n = 2$. Point *ii)* is proved as follows. The first derivative of (45) at $n = 2$ is given by:

$$2\left(p_s - p_n\right)p_n p\mu\left(2p_s - p_n\mu\right)\left(p_s\left(\mu\left(4+\mu\right) - 4\right) - 2p_n\mu^2\right). \tag{50}$$

The sign of (50) is the same as the sign of the following expression:

$$\mu\left(p_s\left(\mu\left(4+\mu\right) - 4\right) - 2p_n\mu^2\right). \tag{51}$$

The expression in brackets in (51) is positive for $p_n < \dfrac{p_s\left(\mu\left(4+\mu\right)-4\right)}{2\mu^2} \equiv \hat{p}_n$. It is directly verifiable that $\hat{p}_n$ is negative for $\mu < \frac{2}{1+\sqrt{2}}$ and positive for $\mu > \frac{2}{1+\sqrt{2}}$.■

We are now ready to prove Proposition 7.

Point *i) ($\mu < \frac{2}{3}$).* we know from Lemma 3 that at $n = 2$ (45) is negative and decreasing in n, and from Lemma 2 we know that (45) is concave in n. This two facts tell us the all points $n \geq 2$ are in the right (and decreasing) branch of the parabola (45). We conclude that (45) is negative for all $n \geq 2$.

Point *ii)* **($\frac{2}{3} < \mu < \frac{2}{1+\sqrt{2}}$).** From Lemma 2 and Lemma 3 we know that (45) is concave and decreasing in n at $n = 2$. These two facts imply that all points $n \geq 2$ are in the right (and decreasing) branch of the parabola (45). In this range of values for μ, however, (45) can be either positive or negative at $n = 2$, depending on the value of p_n (see Lemma 3 point *i)*). Suppose first that (45) is negative at $n = 2$; in this case, the two real roots of (45) are strictly smaller than 2, and (45) remains negative for all $n \geq 2$. Suppose then that (45) is positive at $n = 2$; in this case, the larger real root n_+ must be larger than 2, so that (45) is negative for all $n > n_+$.

Point *iii)* ($\mu > \frac{2}{1+\sqrt{2}}$). In this range, (45) is concave in n if and only if if $p_n > p_n^* > 0$, otherwise it is convex (Lemma 2). Moreover, we know from Lemma 3 that at $n = 2$ (45) is positive. Consider first the case $p_n > p_n^*$ ((45) concave). Here, the larger real root n_+ must be larger than 2, so that for all $n > n_+$ (45) is negative. Consider then the case $p_n < p_n^*$ ((45) convex). Here, at $n = 2$ (45) is increasing in n if $p_n < \hat{p}_n$. Since $\hat{p}_n = p_n^*$ for $\mu = \frac{2}{1+\sqrt{2}}$ and for $\mu > \frac{2}{1+\sqrt{2}}$ the difference $\hat{p}_n - p_n^*$ is increasing in μ [2], it follows that $p_n^* < \hat{p}_n$ for all $\mu > \frac{2}{1+\sqrt{2}}$ and that (45) is increasing in n at $n = 2$.

[2] More precisely, the first derivative of the expression $\frac{p_s\left(\mu\left(4+\mu\right)-4\right)}{2\mu^2} - p_n$ is increasing in μ.

Since in this case (45) is convex, we conclude that the two real roots (n_-, n_+) are smaller than 2, and that (45) is positive for $n \geq 2$. ■

Proof of Proposition 8 We replace the parameters in (1) with those specific for the Beauty Contest: $\gamma_a = -1$; $\gamma_\theta = -v$; $\gamma_{a\theta} = 2v$; $\gamma_A = -\frac{(1-v)}{(n-1)^2}$; $\gamma_{aA} = \frac{2(1-v)}{n-1}$.

Point 1. We study again the difference in expected profits of two firms, 1 and 2, in the complete networks g^c and in the network $g^{-12} \equiv \{g^c - 12\}$. The updating coefficient in g^c is:

$$k_1^{ig^c} = \frac{t}{p_s + (n-1)\, p_n}, \tag{52}$$

from which we obtain the following common equiibrium coefficient:

$$\beta^{g^c} = -\frac{t}{(p_s + (n-1)\, p_n)}.$$

For $g^{-12} \equiv \{g^c - 12\}$, the updating coefficients are:

$$\begin{aligned} k_1^{ig^{-12}} &= \frac{t}{p_s + (n-2)\, p_n}, \quad k_2^i = \frac{p_n}{p_s + (n-2)\, p_n}, \quad i = 1, 2 \\ k_1^{ig^{-12}} &= \frac{t}{p_s + (n-1)\, p_n}, \quad \forall i \geq 3 \end{aligned} \tag{53}$$

We obtain the following equilibrium coefficients for firms 1 and 2 and for firms $j, k > 2$:

$$\begin{aligned} \beta_{11}^{g^{-12}} &= \beta_{22}^{g^{-12}} = -\frac{t(n-1)v}{p_s + (n-2)p_s v + p_n(n-3+(5+(n-4)n)v)} \\ \beta_{1j}^{g^{-12}} &= \beta_{2j}^{g^{-12}} = \\ &= -\frac{t + (n-2)tv}{p_s + (n-2)p_s v + p_n(n-3+(5+(n-4)n)v)}, \\ \beta_{j1}^{g^{-12}} &= \beta_{j2}^{g^{-12}} = \\ &= -\frac{(n-1)((n-2)p_n + p_s)tv}{((n-1)p_n + p_s)(p_s + (n-2)p_s v + p_n(n-3+(5+(n-4)n)v))} \\ \beta_{jj}^{g^{-12}} &= \beta_{jk}^{g^{-12}} = \\ &= -\frac{t(p_s + (n-2)p_s v + (n-1)p_n(1+(n-3)v))}{((n-1)p_n + p_s)(p_s + (n-2)p_s v + p_n(n-3+(5+(n-4)n)v))} \end{aligned}$$

We will now write down the change in expectd payoff of agent 1 moving from g^c to g^{-12} following condition (17-19) and (21). The terms used to compute the other players' aggregate volatility are given by:

$$B_{1k}^{g^c} = (n-1)\beta_{11}^{g^c}; \tag{54}$$
$$B_{11}^{g^{-12}} = (n-2)\beta_{31}^{g^{-12}}; \tag{55}$$
$$B_{12}^{g^{-12}} = \beta_{11}^{g^{-12}} + (n-2)\beta_{31}^{g^{-12}}; \tag{56}$$
$$B_{1k}^{g^{-12}} = \beta_{13}^{g^{-12}} + (n-2)\beta_{33}^{g^{-12}}. \tag{57}$$

From (21) we express the change in payoff moving from g^{-12} to g^c as follows:

$$-\gamma_a[(p_s n + n(n-1)p_n)(\beta_{11}^{g^c})^2 - p_s(\beta_{11}^{g^{-12}})^2 - (n-2)(\beta_{1k}^{g^{-12}})^2 p_s - (2(n-2)\beta_{11}^{g^{-12}}\beta_{1k}^{g^{-12}} + (n-2)(n-3)(\beta_{1k}^{g^{-12}})^2)p_n] + \gamma_A[np_s(B_{21}^{g^c})^2 + n(n-1)(B_{ik}^{g^c})^2 p_n - p_s((B_{12}^{g^{-12}})^2 + (B_{11}^{g^{-12}})^2 + (n-2)(B_{13}^{g^{-12}})^2) - 2p_n((B_{11}^{g^{-12}}(B_{12}^{g^{-12}} + (n-2)B_{13}^{g^{-12}}) + B_{12}^{g^{-12}}(n-2)B_{13}^{g^{-12}} + (B_{13}^{g^{-12}})^2(n-2)(n-3)/2)]$$

Now we show that this expression is never negative for all values of v, p_n and p_s in the ranges $0 < v \leq 1$ and $p_n < p_s$. Replacing the coefficients we get the following expression:

$$\frac{(p_s - p_n)\, t^2 v \left(p_n \left(2n + 12v - 6 + n \cdot v \left(n\left(n-2\right) - 4\right) + v^2 \left(n\left(7-2n\right) - 8\right)\right) + p_s \left(2 + v\left(n^2 - 4 + 3v - 2nv\right)\right)\right)}{\left(\left(n-1\right)p_n + p_s\right)\left(p_s + \left(n-2\right)p_s v + p_n\left(n - 3 + \left(5 + \left(n-4\right)n\right)v\right)\right)^2} \tag{58}$$

It can be shown that the denominator of (58) is strictly positive. Then the sign of (58) is therefore the sign of its numerator, which can be written in the following form:

$$a \cdot v^2 + b \cdot v + c \tag{59}$$

where

$$a = p_n\left(n\left(7-2n\right) - 8\right) + p_s\left(3 - 2n\right)$$
$$b = 12p_n + n \cdot p_n\left(n\left(n-2\right) - 4\right) + p_s\left(n^2 - 4\right)$$
$$c = 2p_s + p_n\left(2n - 6\right)$$

The proof continues now by studying the sign of (59).

We first note that the roots (n_-, n_+) of (59) are real (since $b^2 - 4ac \geq 0$), distinct and finite (since $a \neq 0$). Moreover by a direct inspection of (59) we see that it is concave and that its smaller root is negative and the larger one is greater than 1. Then (58) is positive for all parameter values, implying that the complete network is always stable.

Point 2: We study the difference in expected profits of two agents, 1 and 2, in the empty network $g^{\varnothing}$ and in the network $g^{12} \equiv \{12\}$. The updating coefficients for $g^{\varnothing}$ are:

$$k_1^{ig^{\varnothing}} = \frac{t}{p_s}, \quad k_2^{ig^{\varnothing}} = \frac{p_n}{p_s}, \quad \forall i \tag{60}$$

from which we obtain the common coefficient of agents' equilibrium strategies:

$$\beta^{g^{\varnothing}} = -\frac{tv}{ps + pn(v-1)}.$$

and

$$B_{ii}^{g^{\varnothing}} = 0 \tag{61}$$

$$B_{ij}^{g^{\varnothing}} = \beta^{g^{\varnothing}} \text{ for } i \neq j \tag{62}$$

The updating coefficients for $g^{12} \equiv \{12\}$ are:

$k_1^{ig^{12}} = \frac{t}{p_s+p_n}$ for $i = 1, 2$
$k_1^{ig^{12}} = \frac{t}{p_s}$ for all $i \geq 3$
$k_2^{ig^{12}} = \frac{p_n}{p_s+p_n}$ for $i = 1, 2$
$k_2^{ig^{12}} = \frac{p_n}{p_s}$ for $i \geq 3$

We obtain the following equilibrium coefficients:

$$\beta_{11}^{g^{12}} = \beta_{12}^{g^{12}} = \beta_{21}^{g^{12}} = \beta_{22}^{g^{12}} =$$
$$= -\frac{(n-1)tv(p_n(1-v)+(n-1)p_s)}{(n-1)p_s^2(n+v-2)+p_n^2(v-1)(n(n-1-3v)+5v-2)+p_np_s(n-2+v)(2+(n-3)v)};$$
$$\beta_{kk}^{g^{12}} = \frac{(n-1)tv(p_n(2+n-3v)+p_s(n-2+v))}{(n-1)p_s^2(n-2+v)+p_n^2(v-1)(n-2(n-1-3v)+5v)+p_np_s(n-2+v)(2+(n-3)v)},$$

and:

$$B_{11}^{g^{12}} = \beta_{21}^{g^{12}} \tag{63}$$

$$B_{1k}^{g^{12}} = \beta_{kk}^{g^{12}}. \tag{64}$$

Using (20) we can express the difference in profits of agent 1 (and, by symmetry, of agent 2) in $g^{\varnothing}$ and in g^{12} as:

$$\begin{aligned}
&-\gamma_a[(\beta_{ii}^{g^{\varnothing}})^2 p_s - 2(\beta_{11}^{g^{12}})^2(p_s+p_n)] \\
&\quad + \gamma_A[(n-1)B_{ij}^{g^{\varnothing}\,2} p_s + (n-1)(n-2)B_{ij}^{g^{\varnothing}\,2} p_n - (2(B_{11}^{g^{12}})^2 + (n-2)(B_{1k}^{g^{12}})^2)p_s - \\
&\qquad (2(B_{11}^{g^{12}})^2 + (4n-8)B_{11}^{g^{12}}B_{1k}^{g^{12}} + (n-2)(n-3)(B_{1k}^{g^{12}})^2)p_n]
\end{aligned} \tag{65}$$

The proof then goes through the following steps (complete proofs from authors upon request): we plug in the coefficients' expressions, so we get an expression in n, p_s, p_n and v. Then we find that: i) (65) is strictly positive for $p_n = 0$ for all parameters' values; ii) (65) is equal zero for $p_n = p_s$ for all parameters' values; iii) the derivative of (65) respect to p_n computed in $p_n = p_s$ is strictly negative for all parameters' values; iv) (65) never is negative for $p_n \in (0, p_s)$ and for all other parameters' value. All these evidences are enough to say that (65) is positive for all parameters' values and, consequently, the empty network is not pairwise stable.

Proof of Proposition 9

We replace the parameters in (1) with those specific for the Public Good Game: $\gamma_a = -b$; $\gamma_{A\theta} = 1$; $\gamma_{a\theta} = 1$. Expression (6) becomes:

$$\beta_{ih}^{g} = \frac{t}{2b(p_s + (n_i^g - 1)\, p_n)} \forall h \in N_i; \tag{66}$$

Note as the coefficient that agent i applies to signal h depends only on his degree.

The incentives to form a new link are given by (20) which, when ($g' = g + ij$), is rewritten as follows:

$$\left[cov\left(A_i^{g'}, \theta\right) - cov\left(A_i^g, \theta\right)\right] + b \cdot \left[var\left(a_i^{g'}\right) - var\left(a_i^g\right)\right] \tag{67}$$

Note that the term inside the first brackets depends only on the covariance between the action of agent j and θ. Therefore using (66) and (18), and noting that passing from g to g' only agents i's and j's coefficients change, we can write (67) as:

$$\frac{t^2}{2b}\left[\frac{n_j + 1}{(p_s + n_j p_n)} - \frac{n_j}{(p_s + (n_j - 1)p_n)}\right] + \frac{t^2}{4b} \cdot \left[\frac{n_i + 1}{p_s + n_i p_n} - \frac{n_i}{p_s + (n_i - 1)p_n}\right] \tag{68}$$

that, from a direct inspection, is strictly positive as long as $p_s > p_n$. Therefore every incomplete network is not stable resulting in the complete network being the unique pairwise stable architecture.

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